ZEROS OF A FAMILY OF APPROXIMATIONS OF HECKE L-FUNCTIONS ASSOCIATED WITH CUSP FORMS

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In memory of Professor Marvin Isadore Knopp

Abstract. We consider a family of approximations of a Hecke L-function $L_f(s)$ attached to a holomorphic cusp form f of positive integral weight k with respect to the full modular group. These families are of the form

$$L_f(X;s) := \sum_{n\leq X} \frac{a(n)}{n^s} + (-1)^{k/2}(2\pi)^{-(1-2s)} \frac{\Gamma\left(\frac{k+1}{2}-s\right)}{\Gamma\left(\frac{k-1}{2}+s\right)} \sum_{n\leq X} \frac{a(n)}{n^{1-s}},$$

where $s=\sigma+it$ is a complex variable and $a(n)$ is a normalized Fourier coeffient of f. From an approximate functional equation one sees that $L_f(X;s)$ is a good approximation to $L_f(s)$ when $X=t/2\pi$. We obtain vertical strips where most of the zeros of $L_f(X;s)$ lie. We study the distribution of zeros of $L_f(X;s)$ when X is independent of t. For $X=1$ and 2 we prove that all the complex zeros of $L_f(X;s)$ lie on the critical line $\sigma=1/2$. We also show that as $T\to\infty$ and $X=T^{o(1)}$, 100% of the complex zeros of $L_f(X;s)$ up to height T lie on the critical line. Here by 100% we mean that the ratio between the number of simple zeros on the critical line and the total number of zeros up to height T approaches 1 as $T\to\infty$.

1. Introduction

Let $N\geq 1$ be an integer. Define

$$F_N(s) := \sum_{n\leq N} n^{-s} \quad \text{and} \quad \zeta_N(s) := F_N(s) + \chi(s)F_n(1-s),$$

where $\chi(s)=\pi^{s-1/2}\Gamma((1-s)/2)/\Gamma(s/2)$. Spira [18, 19] appears to be the first author who considered the functions $\zeta_N(s)$ and investigated the zeros of these functions. The behavior of the functions $\zeta_N(s)$ is not completely unknown. From an approximate functional equation we have

$$\zeta(s) = \zeta_N(s) + O(|t|^{-\sigma/2}),$$

where $s=\sigma+it$, $|t|\geq 1$, $|\sigma-1/2|\leq 1/2$, and $N=\sqrt{|t|/2\pi}$ (see Titchmarsh[20]). In [18], Spira proved that all the complex zeros of $\zeta_1(s)$ and $\zeta_2(s)$ lie on the line $\sigma=1/2$. In [19], he presented a numerical computation which suggests that infinitely many zeros are off the line $\sigma=1/2$ for $N\geq 3$. In the same paper, based on numerical evidence, he suggested the following:

> The zeros within the critical strip appear to lie outside the t range $\sqrt{2\pi eN}\leq t\leq 2\pi eN$ for each N. There is also a second, less obvious, t range free of zeros, corresponding to where the Riemann- Siegel formula is used, $N\leq$

2010 *Mathematics Subject Classification.* Primary 11M41, 11F99.

Keywords and phrases. Hecke L-functions, approximate functional equation, proportion of zeros on the critical line.

> $(t/2\pi)^{1/2} < N+1$. In this second region, $g_N(s)$ approximates $\zeta(s)$, while in the first region, $g_N(s)$ is approximately $2\zeta(s)$...

Here $\zeta_N(s) = g_N(s)$. Since then very few related results have appeared in the literature. Very recently, Gonek and Montgomery [10] studied thoroughly the zero distribution of $\zeta_N(s)$. First they provided a proof of Spira's aforementioned claim. In the same paper, Gonek and Montgomery found a zero free region for $\zeta_N(s)$ and also obtained further results on the numbers of zeros of $\zeta_N(s)$. They proved the striking result that 100% of the complex zeros of $\zeta_N(s)$ lie on the critical line, provided N is not too large with respect to the height T. We will discuss this fact later.

Gonek and Ledoan [9], Langer [13], and Wilder [23] proved asymptotic results for the number of zeros of $F_N(s)$. If $N_F(T)$ is the number of zeros of $F_N(T)$ up to height T, then they found that

$$N_F(T) = \frac{T}{2\pi} \log X + O(X).$$

This result is an indispensable ingredient to obtain good lower bound for the number of zeros of $\zeta_N(s)$ on the critical line. In fact the growth rate of the error term offers a comparison between the growth rate of the number of zeros on the critical line up to height T and the total number of zeros of $\zeta_N(s)$ up to height T. It is worthy to mention that, in [14], Ledoan and last two authors presented some instances where the error term can be improved.

Let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ be the full modular group. Let $f \in S_k(\Gamma)$ be a holomorphic cusp form of even integral weight $k > 0$ for Γ, with Fourier series given by

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}.$$

We also assume that f is a normalized primitive Hecke form with $a_f(1) = 1$. Let $a(n) := a_f(n) n^{(1-k)/2}$ and let $L_f(s)$ be the L-function associated to f, defined by

$$L_f(s) := \sum_{n=1}^{\infty} a(n) n^{-s}, \tag{1.1}$$

for $\operatorname{Re} s > 1$.

In [12], Knopp, Kohnen, and Pribitkin studied the sign changes of the Fourier coefficients $a(n)$ of a cusp form f for $\mathrm{SL}(2,\mathbb{R})$. They showed that these coefficients $a(n)$ change sign infinitely often. For more about the coefficients $a(n)$, one may consult the monograph of Berndt and Knopp [4].

Next, we consider the partial sums

$$\sum_{n \le X} \frac{a(n)}{n^s}.$$

Let $N(X;T)$ denote the number of complex zeros of $\sum_{n \le X} a(n) n^{-s}$ up to height T. Then as a special case of Theorem 3 in [13], one obtains the following result.

Proposition 1.1. *Let M be the largest integer less than or equal to X such that $a(M) \neq 0$. Then we have*

$$N(X;T) = \frac{T}{2\pi} \log M + O_f(X).$$

From Deligne's proof [6, 7] of the Ramanujan-Peterson conjecture, which is a consequence of the Riemann Hypothesis for varieties over finite fields, the coefficients $a(n)$ satisfy the bound

$$|a(n)| \le d(n), \tag{1.2}$$

where $d(n)$ is the divisor function. In particular

$$|a(p)| \le 2, \tag{1.3}$$

for all primes p. The divisor function satisfies [1, p. 296]

$$d(n) \le c_\delta n^\delta \le n, \tag{1.4}$$

for any $\delta > 0$, and moreover by a result of Wigert [22],

$$\log(d(n)) \le \frac{\log 2 \log n}{\log\log n} + O\left(\frac{\log n}{(\log\log n)^2}\right).$$

The L-function $L_f(s)$ has an analytic continuation throughout the complex plane as an entire function, by

$$(2\pi)^{-s-\frac{k-1}{2}}\Gamma\left(s+\tfrac{k-1}{2}\right)L_f(s) = \int_0^\infty f(iy)y^{s+\frac{k-1}{2}-1}\,dy,$$

and it satisfies the functional equation

$$L_f(s) = \chi_f(s)L_f(1-s), \tag{1.5}$$

where

$$\chi_f(s) := (-1)^{k/2}(2\pi)^{-(1-2s)}\frac{\Gamma\left(\frac{k+1}{2}-s\right)}{\Gamma\left(\frac{k-1}{2}+s\right)}. \tag{1.6}$$

A straightforward computation shows that

$$\chi_f(s)\chi_f(1-s) = 1. \tag{1.7}$$

The Euler product representation of $L_f(s)$ is

$$L_f(s) = \prod_p \left(1 - a(p)p^{-s} + p^{-2s}\right)^{-1}, \tag{1.8}$$

where $\operatorname{Re} s > 1$. The non-trivial zeros of $L_f(s)$ lie within the critical strip $0 < \operatorname{Re} s < 1$, symmetrically with respect to the real axis and the critical line $\operatorname{Re} s = 1/2$. The Riemann hypothesis for $L_f(s)$ states that, all the non-trivial zeros of $L_f(s)$ lie on the critical line $\operatorname{Re} s = 1/2$.

Let $N_f(T)$ denote the number of non-trivial zeros ρ of $L_f(s)$ for which $0 < \operatorname{Im}\rho < T$, for T not equal to any $\operatorname{Im}\rho$; otherwise we put

$$N_f(T) = \lim_{\epsilon\to 0}\frac{1}{2}\{N_f(T+\epsilon) + N_f(T-\epsilon)\}.$$

Then one can show that [15]

$$N_f(T) = \frac{T}{\pi}\log\frac{T}{2\pi} - \frac{T}{\pi} + O(\log T).$$

An approximate functional equation of $L_f(s)$ (see Apostol and Sklar [2], Chandrasekharan and Narasimhan [5], and Good [11]) is given by

$$L_f(s) = \sum_{n\le X}\frac{a(n)}{n^s} + \chi_f(s)\sum_{n\le X}\frac{a(n)}{n^{1-s}} + O(|t|^{1/2-\sigma+\epsilon}), \tag{1.9}$$

for $\epsilon > 0, |t| \gg 1, |\sigma - 1/2| \le 1/2$ and $X = \frac{|t|}{2\pi}$. Let us define

$$L_f(N;s) := \sum_{n\le N} \frac{a(n)}{n^s} + \chi_f(s) \sum_{n\le N} \frac{a(n)}{n^{1-s}}. \tag{1.10}$$

From (1.6) and (1.10), we have the following functional equation,

$$L_f(N;s) = \chi_f(s) L_f(N;1-s). \tag{1.11}$$

Since $f \in S_k(\Gamma)$ is a primitive Hecke form, then all $a(n) \in \mathbb{R}$. Therefore $L_f(N;s)$ is real for all real values of s. So the zeros of $L_f(N;s)$ are symmetric with respect to the real axis. Also from the functional equation (1.11) we find that the zeros of $L_f(N;s)$ are symmetric with respect to the critical line $\sigma = 1/2$. By a generalization of Descartes's Rule of Signs (see Pólya and Szegö [16], Part V, Chapter 1, No. 77), $\sum_{n\le N} a(n)n^{-s}$ has at most finitely many real roots for real values of s. Also from (1.6), $\chi_f(s)$ has simple poles at all half-integers greater than or equal to $(k+1)/2$. Therefore there exists a real number α, so that all half-integers greater than α are simple poles of $L_f(N;s)$. Hence $L_f(N;s)$ is analytic everywhere except possibly for simple poles at half-integers.

From (1.9) and (1.11), we observe that $L_f(N;s)$ approximates $L_f(s)$ for $N < \frac{|t|}{2\pi} < N+1$, except possibly at the critical line. From [2, Theorem 2] we have

$$L_f(s) = \sum_{n\le N} \frac{a(n)}{n^s} + O(N^{1/4-\sigma}), \tag{1.12}$$

uniformly for $\sigma \ge \sigma_1 > -1/4$, provided $N > B\left(\frac{t}{4\pi}\right)^2$ for some $B > 1$. From Stirling's formula we know that in vertical strips,

$$|\chi_f(s)| = \left(\frac{|t|}{2\pi e}\right)^{1-2\sigma} \left(1 + O_f\left(\frac{1}{|t|}\right)\right), \tag{1.13}$$

$|t| \to \infty$ (see (6.8) for a proof). From (1.10), (1.12), (1.7), and (1.13) we find that

$$L_f(N;s) = 2L_f(s) + O(N^{1/4-\sigma}) + O(|t|^{1-2\sigma} N^{\sigma-3/4}), \tag{1.14}$$

uniformly for $\min(\sigma, 1-\sigma) \ge \sigma_1 > -1/4$, provided $N > B\left(\frac{t}{4\pi}\right)^2$ for some $B > 1$. Since $|t| \ll \sqrt{N}$, the error terms in (1.14) are $\ll |t|^{-\min(1/2, 2\sigma-1/2)}$, uniformly for $1/4 < \sigma < 3/4$. Hence

$$L_f(N;s) = 2L_f(s) + O(|t|^{-\min(1/2,\sigma-1/4)}), \tag{1.15}$$

uniformly for $1/4 < \sigma < 3/4$ and $|t| \ll \sqrt{N}$. This shows that $L_f(N;s)$ approximates $2L_f(s)$ near the critical line for sufficiently large t in the range $|t| \ll \sqrt{N}$.

For Re $s > 1$, let

$$L_\tau(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s},$$

where $\tau(n)$ is the Ramanujan τ-function. In [3], Berndt obtained the inequality

$$|L_\tau(12-s)| > |L_\tau(s)|,$$

for $|t| \ge 6.8$ and when $L_\tau(s) \ne 0$. In [17], Spira proved the same inequality but improved the bound to $|t| \ge 4.35$. Very recently in [21], Trudgian improved this bound for t to $|t| \ge 3.8085$. In this article, we show that a similar inequality also holds for $L_f(N;s)$. We have the following theorem.

Theorem 1.2. *Let N be a positive integer. Then the inequality $|L_f(N;1-s)| > |L_f(N;s)|$ holds for all s with $t > t_k$ and $1/2 < \sigma < 1$, if and only if all the zeros $\beta + i\gamma$ of $L_f(N;s)$ with $\beta \in (0,1)$ and $\gamma > t_k$ lie on the critical line. Here t_k is a real number depending on the weight k of the cusp form f. In particular, $t_{12} = 3.8027$, $t_{14} = 1.8477$, and $t_k = 0$ for $k \geq 16$.*

As with the results in [18], one can prove that the non-trivial zeros of $L_f(1;s)$ and $L_f(2;s)$ lie on the critical line. In the case of primitive Hecke forms the coefficients could be as big as the divisor function $d(n)$, and we will prove our theorem for some restricted primitive Hecke forms.

Theorem 1.3. *All the zeros of $L_f(1;s)$ with $|t| > \max(k, e^{16})$ lie on the critical line. Moreover, if $|a(2)| \leq 1$ then all the zeros of $L_f(2;s)$ with $|t| > \max(k, e^{16})$ also lie on the critical line.*

Remark: Numerical computation shows that $a(2) = \tau(2)2^{-11/2} = -.53033$, thus the L-function attached to the Ramanujan τ-function satisfies the above theorem.

We are interested to see whether for $N \geq 3$, the non-trivial zeros of $L_f(N;s)$ lie on the critical line or not. Although it is not clear whether all the non-trivial zeros of $L_f(N;s)$ for $N \geq 3$ lie on the critical line or not, one can prove that a positive proportion of the non-trivial zeros of $L_f(N;s)$ lie on the critical line, provided N is not too large relative to the height T of the ordinates of the non-trivial zeros.

In the following theorem we obtain a 'critical' strip for $L_f(N;s)$. More precisely,

Theorem 1.4. *Let $\lambda > 1/2$. There exists a constant N_0 such that if $N \geq N_0$ and $\beta + i\gamma$ is a zero of $L_f(N;s)$ with $|\gamma| \geq 2\pi e N^\lambda$, then*

$$\big|\beta - 1/2\big| \leq \begin{cases} \frac{1}{2\lambda-1}\big(\frac{1}{2} + \frac{4\lambda \log\log N}{\log N}\big), & \text{if } 1/2 < \lambda \leq 1 \\ \frac{1}{2} + \frac{4 \log\log N}{\log N}, & \text{if } \lambda \geq 1. \end{cases} \tag{1.16}$$

One also obtains a critical strip for $N \leq N_0$, provided that the ordinates of the zeros are sufficiently large. We have

Theorem 1.5. *There exists a constant T_0 such that if $N \geq 1$ and $\beta + i\gamma$ is a zero of $L_f(N;s)$ with $|\gamma| \geq \max(2\pi e N, T_0)$, then*

$$\big|\beta - 1/2\big| \leq 3.$$

Next we will estimate the number of zeros of $L_f(N;s)$ where the ordinates of the zeros lie in an interval of the form $(T, T+U]$. We define

$$N(T) = \#\{\rho = \beta + i\gamma : 0 < \gamma < T \text{ and } L_f(N;\rho) = 0\}$$

and

$$N^0(T) = \#\{\rho = 1/2 + i\gamma : 0 < \gamma < T \text{ and } L_f(N;\rho) = 0\}.$$

We have the following theorem.

Theorem 1.6. *Let $\lambda > 1/2$. There exists a constant N_0 such that if $N > N_0$, $T > 2\pi e N^\lambda$ and $U \geq 2$, then*

$$N^0(T+U) - N^0(T) \geq N(T+U) - N(T) + O_f(U \log N) + O_f(N) + O_f\left(\left(\frac{\lambda}{2\lambda - 1}\right)^3 \log(T+U)\right). \tag{1.17}$$

Furthermore there exists a constant T_0 such that if $N \geq 1$ and $T > \max(2\pi e N, T_0)$ then (1.17) holds with the last error term replaced by $O_f(\log(T+U))$.

We end the introduction with the following result.

Theorem 1.7. *As $T \to \infty$ and $N = T^{o(1)}$, 100% of the non-trivial zeros of $L_f(N;s)$ up to height T are simple and lie on the critical line.*

A natural question that we pose to interested readers would be to find an appropriate axiomatic context where one can treat the problems discussed above (which may or may not be the same for each of the above theorems). In what follows we restrict ourselves to the context presented in this introduction and derive the very concrete results stated above.

2. Preliminary Results

The following lemmas which may be of independent interest are instrumental in the proof of the theorems.

Lemma 2.1. *For $\sigma > 1$ we have*

$$\left(\frac{\sigma-1}{\sigma}\right)^2 < |L_f(s)| < \left(\frac{\sigma}{\sigma-1}\right)^2. \tag{2.1}$$

Proof. Let $\sigma > 1$. From (1.1) and (1.2) we have

$$|L_f(s)| \le \sum_{n=1}^{\infty} \frac{|a(n)|}{n^\sigma} \le \sum_{n=1}^{\infty} \frac{d(n)}{n^\sigma} = \left(\sum_{n=1}^{\infty} \frac{1}{n^\sigma}\right)^2 < \left(1 + \int_1^\infty x^{-\sigma}\,dx\right)^2 = \left(\frac{\sigma}{\sigma-1}\right)^2. \tag{2.2}$$

For the other inequality in (2.1) we use the Euler product (1.8). From (1.2) we have

$$|L_f(s)| = \prod_p |\left(1 - a(p)p^{-s} + p^{-2s}\right)|^{-1} \ge \prod_p \left(1 + d(p)p^{-\sigma} + p^{-2\sigma}\right)^{-1}.$$

Since $d(p) = 2$, we find that

$$|L_f(s)| \ge \prod_p \left(1 + 2p^{-\sigma} + p^{-2\sigma}\right)^{-1} = \prod_p \left(1 + p^{-\sigma}\right)^{-2} = \left(\frac{\zeta(2\sigma)}{\zeta(\sigma)}\right)^2 > \left(\sum_{n=1}^{\infty} \frac{1}{n^\sigma}\right)^{-2}$$
$$> \left(\frac{\sigma-1}{\sigma}\right)^2,$$

where in the ultimate step we used the last three inequalities in (2.2). This completes the proof of the lemma. □

Lemma 2.2. *For $\sigma > 1$,*

$$\left|\sum_{n>N} \frac{a(n)}{n^s}\right| \le \frac{N^{1-\sigma}}{\sigma-1}\left(\log N + 2\gamma + \frac{1}{\sigma-1}\right) + O\left(\frac{1}{\sqrt{N}}\right). \tag{2.3}$$

For $\sigma \le 0$ we have the following:

$$\left|\sum_{n\le N} \frac{a(n)}{n^s}\right| \le N^{1-\sigma}(\log N + 2\gamma - 1) + O(N^{-\sigma+1/2}). \tag{2.4}$$

Proof. Let $\sigma > 1$. From (1.2) and by partial summation we have

$$\left|\sum_{n>N} \frac{a(n)}{n^s}\right| \le \sum_{n>N} \frac{d(n)}{n^\sigma} = \sigma \int_N^\infty D(t)t^{-1-\sigma}\,dt - D(N)N^{-\sigma}, \tag{2.5}$$

where

$$D(t) = \sum_{n\le t} d(n) = t(\log t + 2\gamma - 1) + O(\sqrt{t}). \tag{2.6}$$

Combining (2.5) and (2.6) we obtain the bound in (2.3).

For the second part of the lemma, let $\sigma \le 0$. We have

$$\left|\sum_{n\le N} \frac{a(n)}{n^s}\right| \le \sum_{n\le N} \frac{d(n)}{n^\sigma} \tag{2.7}$$

By using (2.6) one sees that

$$\sum_{n\le N} \frac{d(n)}{n^\sigma} \le N^{-\sigma} \sum_{n\le N} d(n) = N^{1-\sigma}(\log N + 2\gamma - 1) + O(N^{-\sigma+1/2}), \tag{2.8}$$

where in the penultimate step we use the fact that $x^{-\sigma}$ is increasing for $\sigma \le 0$. One finishes the proof of the lemma by combining (2.7) and (2.8). □

Lemma 2.3. *If $|t| > k$ and $1/2 < \sigma < (k-1)/2$ then*

$$\frac{\partial}{\partial\sigma}\left(\log \frac{1}{|\chi_f(s)|}\right) > 2\log|t| - 3.7.$$

Proof. By Stirling's formula [8], we have

$$\log\Gamma(s) = (s-1/2)\log s - s + \frac{1}{2}\log 2\pi + \frac{1}{12s} - 2\int_0^\infty \frac{P_3(x)}{(s+x)^3}\,dx, \tag{2.9}$$

where $P_3(x)$ is a function of period 1 and given by

$$P_3(x) = \frac{x}{12}(2x^2 - 3x + 1),$$

for $x \in [0,1]$. A straightforward computation shows that

$$|6P_3(x)| \le \frac{\sqrt{3}}{36}, \tag{2.10}$$

for $x \in [0,1]$. Since

$$\frac{\partial}{\partial\sigma}\left(\log \frac{1}{|\chi_f(s)|}\right) = -\operatorname{Re}\left(\frac{\partial}{\partial\sigma}\log\chi_f(s)\right) = -\operatorname{Re}\left(\frac{\partial}{\partial s}\log\chi_f(s)\right),$$

then from (1.6) and (2.9) we find

$$\begin{aligned}\frac{\partial}{\partial\sigma}\left(\log \frac{1}{|\chi_f(s)|}\right) = \operatorname{Re}\Bigg(&-\frac{1}{2s+k-1} - \frac{1}{3(k+2s-1)^2} + \frac{1}{2s-k-1} - \frac{1}{3(k-2s+1)^2}\\ &+\log\left(\frac{k-1}{2}+s\right) + \log\left(\frac{k+1}{2}-s\right) - 2\log(2\pi)\\ &+6\int_0^\infty \frac{P_3(x)}{(s+(k-1)/2+x)^4}\,dx + 6\int_0^\infty \frac{P_3(x)}{((k+1)/2-s+x)^4}\,dx\Bigg).\end{aligned} \tag{2.11}$$

From the hypothesis we have $t > k$ and $1/2 < \sigma < (k-1)/2$. Then from (2.11) we derive

$$\frac{\partial}{\partial\sigma}\left(\log \frac{1}{|\chi_f(s)|}\right) > 2\log|t| - 2\log 2\pi - \frac{k}{t^2} - \frac{\sqrt{3}\pi}{72|t|^3} > 2\log|t| - 3.7.$$

Here we use the fact that $k \geq 12$. This proves the lemma. □

Lemma 2.4. *If $|t| > 20$ and $\sigma > 1/2$ then*

$$|\chi_f(s)| < 1.02 \left(\frac{|s|}{2\pi e}\right)^{1-2\sigma}.$$

Proof. From [18, 17], we have

$$|\Gamma(s)| = (2\pi)^{1/2} e^{-\sigma} |s|^{\sigma-1/2} e^{-t \arg s} |\exp(R_1(s) + 1/(12s))|, \tag{2.12}$$

where $R_1(s) < 1/(6|s|)$. Hence by (1.6) and (2.12) we find

$$\begin{aligned} |\chi_f(s)| = \left(\frac{|s|}{2\pi e}\right)^{1-2\sigma} & \exp(t(\arg((k+1)/2 - s) + \arg((k-1)/2 + s)))) \times \\ & \frac{\left|1 - \frac{k+1}{2s}\right|^{k/2-\sigma}}{\left|1 + \frac{k-1}{2s}\right|^{(k-2)/2+\sigma}} \frac{|\exp(R_1((k+1)/2 - s) + 1/12((k+1)/2 - s))|}{|\exp(R_1(((k-1)/2 + s)) + 1/12((k-1)/2 + s))|}. \end{aligned} \tag{2.13}$$

Next we set

$$z = R_1\left(\frac{k-1}{2} + s\right) - R_1\left(\frac{k+1}{2} - s\right) + \frac{1}{12\left(\frac{k-1}{2} + s\right)} - \frac{1}{12\left(\frac{k+1}{2} - s\right)}.$$

Therefore

$$|z| \leq \frac{1}{12\left|\frac{k-1}{2} + s\right|} + \frac{1}{12\left|\frac{k+1}{2} - s\right|} + \frac{1}{6\left|\frac{k-1}{2} + s\right|} + \frac{1}{6\left|\frac{k+1}{2} - s\right|} \leq \frac{1}{2|t|} \leq \frac{1}{40}.$$

Since $|z| \leq 1/40 < 1$, we have

$$|e^z| \geq 1 - |z| \left(\frac{1}{1 - |z|}\right) \geq 38/39. \tag{2.14}$$

Clearly

$$t\left(\arg\left(\frac{k+1}{2} - s\right) + \arg\left(s + \frac{k-1}{2}\right)\right) < 0. \tag{2.15}$$

Combining (2.13), (2.14), and (2.15), we obtain

$$|\chi_f(s)| < 1.02 \left(\frac{|s|}{2\pi e}\right)^{1-2\sigma},$$

which proves the lemma. □

3. Proof of Theorem 1.2

We first prove the following theorem.

Theorem 3.1. *There exists a number t_k, such that for $1/2 < \sigma < 1$ and $|t| > t_k$ we have*

$$|L_f(N; 1-s)| > |L_f(N; s)|,$$

whenever $L_f(N; s) \neq 0$. Moreover the above holds with $t_{12} = 3.8027$, $t_{14} = 1.8477$ and $t_k = 0$ for $t \geq 16$.

Proof. From (1.11) we have

$$L_f(N;1-s)=g(s)L_f(N;s), \tag{3.1}$$

where $g(s)=1/\chi_f(s)$. From (1.6) one can see that $g(s)$ is analytic for all s with $t\neq 0$ and hence continuous for such s. Define $h(s):=\log|g(s)|$. It suffices to prove that $h(s)>0$ for $1/2<\sigma<1$ provided $|t|\geq t_k$. We have

$$\begin{aligned} h(s) &= \log\left|(2\pi)^{-(2s-1-2it)}\frac{\Gamma\left(\frac{k-1}{2}+s\right)}{\Gamma\left(\frac{k+1}{2}-s\right)}\right| \\ &= -(2\sigma-1)\log 2\pi+\log\left|\Gamma\left(\tfrac{k-1}{2}+s\right)\right|-\log\left|\overline{\Gamma\left(\tfrac{k+1}{2}-s\right)}\right| \\ &= -(2\sigma-1)\log 2\pi+\log\left|\Gamma\left(\tfrac{k-1}{2}+\sigma+it\right)\right|-\log\left|\Gamma\left(\tfrac{k+1}{2}-\sigma+it\right)\right| \\ &= -(2\sigma-1)\log 2\pi+(2\sigma-1)\frac{\partial}{\partial\sigma}\log\left|\Gamma(\sigma+it)\right|\big|_{\sigma=\sigma_1}, \end{aligned} \tag{3.2}$$

for some σ_1 between $\frac{k-1}{2}$ and $\frac{k+1}{2}$. Thus it suffices to prove that

$$\frac{\partial}{\partial\sigma}\log\left|\Gamma(\sigma+it)\right|\big|_{\sigma=\sigma_1}-\log 2\pi>0,$$

for all $\frac{k-1}{2}\leq\sigma_1\leq\frac{k+1}{2}$. Now from (2.9) we have

$$\begin{aligned} \frac{\partial}{\partial\sigma}\log|\Gamma(\sigma+it)| &= \frac{\partial}{\partial\sigma}\ \mathrm{Re}\ \log\Gamma(\sigma+it) \\ &= \mathrm{Re}\ \frac{\partial}{\partial\sigma}\log\Gamma(\sigma+it) \\ &= \mathrm{Re}\ \frac{\partial}{\partial s}\log\Gamma(s) \\ &= \mathrm{Re}\ \left(\log s-\frac{1}{2s}-\frac{1}{12s^2}+6\int_0^\infty\frac{P_3(x)}{(s+x)^4}\,dx\right) \\ &= \log\sqrt{\sigma^2+t^2}-\frac{\sigma}{2(\sigma^2+t^2)}-\frac{\sigma^2-t^2}{12(\sigma^2+t^2)^2} \\ &\qquad +6\int_0^\infty\frac{P_3(x)((\sigma+x)^4-6(\sigma+x)^2t^2+t^4)}{\left((\sigma+x)^2+t^2\right)^4}\,dx. \end{aligned} \tag{3.3}$$

Using (2.10) in combination with the inequality $(\sigma+x)^4-6(\sigma+x)^2t^2+t^4\leq\left((\sigma+x)^2+t^2\right)^2$ and (3.3), we derive

$$\begin{aligned} \frac{\partial}{\partial\sigma}\log|\Gamma(\sigma+it)| &\geq \log\sqrt{\sigma^2+t^2}-\frac{\sigma}{2(\sigma^2+t^2)}-\frac{\sigma^2-t^2}{12(\sigma^2+t^2)^2}-\frac{\sqrt{3}}{36}\int_0^\infty\frac{dx}{\left((\sigma+x)^2+t^2\right)^2} \\ &=: G(\sigma)-I(\sigma), \end{aligned} \tag{3.4}$$

where $I(\sigma)$ is the last term and $G(\sigma)$ is the first three terms of (3.4). Here $I(\sigma)$ is a decreasing function of σ and hence

$$I(\sigma)\leq\frac{\sqrt{3}}{36}\int_0^\infty\frac{dx}{\left(\left(x+\frac{k-1}{2}\right)^2+t^2\right)^2}=\frac{\sqrt{3}}{72t^3}\left(\tan^{-1}\left(\frac{2t}{k-1}\right)-\frac{2t(k-1)}{4t^2+(k-1)^2}\right).$$

Next

$$G'(\sigma)=\frac{\sigma^3\left(6\sigma^2+3\sigma+1\right)+3\sigma\left(4\sigma^2-1\right)t^2+(6\sigma-3)t^4}{6\left(\sigma^2+t^2\right)^3},$$

thus $G(\sigma)$ is increasing on $\frac{k-1}{2}\le\sigma\le\frac{k+1}{2}$ for $k\ge 12$. Hence

$$\begin{aligned}\frac{\partial}{\partial\sigma}\log|\Gamma(\sigma+it)|-\log 2\pi &\ge G\left(\frac{k-1}{2}\right)-\frac{\sqrt{3}}{72t^3}\left(\tan^{-1}\left(\frac{2t}{k-1}\right)-\frac{2t(k-1)}{4t^2+(k-1)^2}\right)-\log 2\pi\\ &\ge \frac{4(4-3k)t^2-(k-1)^2(3k-2)}{3\left((k-1)^2+4t^2\right)^2}+\frac{1}{2}\log\left(\frac{1}{4}(k-1)^2+t^2\right)\\ &\qquad -\frac{\sqrt{3}}{72t^3}\left(\tan^{-1}\left(\frac{2t}{k-1}\right)-\frac{2t(k-1)}{4t^2+(k-1)^2}\right)-\log 2\pi\\ &=: H(t,k). \end{aligned} \tag{3.5}$$

Let us fix $t>0$ and consider k as a real variable for a moment. Then

$$\frac{\partial}{\partial k}H(t,k)=\frac{9(k-2)\left((k-1)^2+4t^2\right)^2+2\left(9(k-3)k+\sqrt{3}+18\right)\left((k-1)^2+4t^2\right)+24(k-1)^3}{9\left((k-1)^2+4t^2\right)^3}\ge 0, \tag{3.6}$$

for $k\ge 12$. Hence for every fixed $t>0$, $H(t,k)$ is monotonically increasing with respect to the variable k. Next let $k\ge 12$ be a fixed number and vary t. Let

$$\begin{aligned}\frac{M(t,k)}{t^4} &:= \frac{\partial}{\partial t}H(t,k)\\ &= \frac{384(3(k-1)k-1)t^6+16(k-1)\left(9k((k-1)k+1)-5\sqrt{3}-9\right)t^4}{36t^3\left((k-1)^2+4t^2\right)^3}\\ &\quad -\frac{32\sqrt{3}(k-1)^3t^2+3\sqrt{3}(k-1)^5-2304t^8}{36t^3\left((k-1)^2+4t^2\right)^3}+\frac{6\sqrt{3}}{144t^4}\tan^{-1}\left(\frac{2t}{k-1}\right). \end{aligned} \tag{3.7}$$

One finds that

$$\begin{aligned}&\frac{\partial}{\partial t}M(t,k)\\ &=\frac{4t^4\left(48\left(33k^2-60k+25\right)t^4+4(k-1)\left(3k\left(39k^2-81k+25\right)+8\sqrt{3}+51\right)t^2\right)}{9\left((k-1)^2+4t^2\right)^4}\\ &\qquad+\frac{4t^4\left((k-1)^3\left(45k((k-1)k+1)+8\sqrt{3}-45\right)+1728t^6\right)}{9\left((k-1)^2+4t^2\right)^4}\\ &\ge 0, \end{aligned} \tag{3.8}$$

for all $t>0$. Therefore combining (3.7), (3.8) and the fact that $M(0,k)=0$, we conclude that $H(t,k)$ is monotonically increasing with respect to t for $t>0$ and fixed $k\ge 12$. One can check that $H(3.8027,12)>0$, $H(1.8477,14)>0$ and $H(t,16)>0$ for all $t>0$, which completes the proof of Theorem 3.1.

□

By the functional equation (1.11), $L_f(N;s)$ and $L_f(N;1-s)$ have the same zeros for $0<\sigma<1$. Hence Theorem 3.1 implies Theorem 1.2.

4. Proof of Theorem 1.3

The proof follows closely the approach from [18]. For the sake of completeness we provide the details below. From (1.10) we have

$$L_f(1;s) = 1 + \chi_f(s). \tag{4.1}$$

Now from the proof of Theorem 3.1 we have for $t > 3.8027$ and $\sigma > 1/2$

$$|\chi_f(s)| < 1. \tag{4.2}$$

Therefore from (4.1) and (4.2) we find that for $t > 3.8027$ and $\sigma > 1/2$,

$$|L_f(1;s)| \geq 1 - |\chi_f(s)| > 0. \tag{4.3}$$

From Theorem 1.7 and Theorem 1.2 we conclude that, all the complex zeros of $L_f(1;s)$ lie on the line $\sigma = 1/2$ for $t > 3.8027$.

Again from (1.10) we see that

$$|L_f(2;s)| \geq \left|1 + \frac{a(2)}{2^s}\right| - |\chi_f(s)| \left|1 + \frac{a(2)}{2^{1-s}}\right|. \tag{4.4}$$

So it suffices to prove that for large enough t and $\sigma > 1/2$,

$$1/|\chi_f(s)| > \left|\frac{1 + \frac{a(2)}{2^{1-s}}}{1 + \frac{a(2)}{2^s}}\right|. \tag{4.5}$$

Let

$$g_1(s) = \chi_f(s)\frac{1 + \frac{a(2)}{2^{1-s}}}{1 + \frac{a(2)}{2^s}}. \tag{4.6}$$

Then $|g_1(1/2 + it)| = 1$. Define

$$l(s) = \log\left|\frac{g_1(s)}{g_1(1/2 + it)}\right|. \tag{4.7}$$

Proceeding as in the proof of Theorem 3.1 one can derive that

$$l(s) = \left(\sigma - \frac{1}{2}\right)\frac{\partial}{\partial\sigma}\left(\log\frac{1}{|\chi_f(s)|} - \log\left|\frac{1 + \frac{a(2)}{2^{1-s}}}{1 + \frac{a(2)}{2^s}}\right|\right)\Bigg|_{\sigma=\sigma_1}, \tag{4.8}$$

for some σ_1 in $[1/2, 1]$. We want to show that

$$\frac{\partial}{\partial\sigma}\left(\log\frac{1}{|\chi_f(s)|}\right)\Bigg|_{\sigma=\sigma_1} > \frac{\partial}{\partial\sigma}\left(\log\left|\frac{1 + \frac{a(2)}{2^{1-s}}}{1 + \frac{a(2)}{2^s}}\right|\right)\Bigg|_{\sigma=\sigma_1}, \tag{4.9}$$

for some $\sigma_1 \in (1/2, 1)$. We distinguish two cases according as to when $1/2 < \sigma \leq 3/4$ and, respectively, when $3/4 < \sigma < 1$. We have

$$\begin{aligned}\frac{\partial}{\partial\sigma}\left(\log\left|\frac{1 + \frac{a(2)}{2^{1-s}}}{1 + \frac{a(2)}{2^s}}\right|\right) &= \operatorname{Re}\frac{\partial}{\partial\sigma}\left(\log\frac{1 + \frac{a(2)}{2^{1-s}}}{1 + \frac{a(2)}{2^s}}\right) \\ &= a(2)\log 2 \operatorname{Re}\left(\frac{a(2) + 2^{s-1} + 2^{-s}}{(1 + a(2)2^{s-1})(1 + a(2)2^{-s})}\right).\end{aligned} \tag{4.10}$$

Then for $1/2 < \sigma \le 3/4$, using (1.3) we have

$$\frac{\partial}{\partial\sigma}\left(\log\left|\frac{1+\frac{a(2)}{2^{1-s}}}{1+\frac{a(2)}{2^s}}\right|\right) \le \log 2\left(\frac{1+2^{\sigma-1}+2^{-\sigma}}{(1-2^{\sigma-1})(1-2^{-\sigma})}\right) < 27. \tag{4.11}$$

Therefore for $1/2 < \sigma \le 3/4$, by Lemma 2.3 and (4.11) we find that the inequality (4.9) holds when $2\log|t| > 27 + 3.7$. In particular one can take $t > e^{16}$. Now consider the case when $3/4 < \sigma < 1$. One can see by (1.3) that

$$\left|\frac{1+\frac{a(2)}{2^{1-s}}}{1+\frac{a(2)}{2^s}}\right| \le \left|\frac{1+2^{\sigma-1}}{1-2^{-\sigma}}\right| \le \frac{1+2}{1-2^{-3/4}} < 5. \tag{4.12}$$

Then from (4.12) and Lemma 2.4, it is enough to show that

$$.98\left(\frac{|s|}{2\pi e}\right)^{2\sigma-1} > 5 \tag{4.13}$$

in order to prove the inequality (4.5). Here (4.13) holds true for $t > 445$. For $\sigma \ge 1$, $1 + 2^{\sigma-1} \le 2^\sigma$ and (4.13) transforms to

$$.98\left(\frac{|s|}{2\sqrt{2}\pi e}\right)^{2\sigma-1} > \frac{\sqrt{2}}{1-2^{-3/4}}. \tag{4.14}$$

Numerical computation shows that $t > 86$ satisfies (4.14) for $\sigma \ge 1$. This completes the proof of the theorem.

5. Proof of Theorems 1.4 and 1.5

Let $\rho_N = \beta_N + i\gamma_N$ be a complex zero of $L_f(N;s)$ with $|\gamma_N| \ge 2\pi e N^\lambda$. We will show that $L_f(N;s)$ never vanishes for

$$\beta_N > \frac{\lambda}{2\lambda-1}\left(1+\frac{4\log\log N}{\log N}\right),$$

when $1/2 < \lambda \le 1$ and is nonzero for

$$\beta_N > 1 + \frac{4\log\log N}{\log N},$$

when $\lambda > 1$. Then one concludes the proof of the theorem by using the functional equation (1.11). Let s be such that $|t| \ge 2\pi e N^\lambda$ with $\lambda > 1/2$ and

$$\sigma > \max\left(1, \frac{\lambda+\epsilon}{2\lambda-1}\right)\left(1+\frac{c\log\log N}{\log N}\right), \tag{5.1}$$

where $\epsilon > 0$ is arbitrary and c is a positive constant which will be determined later. From (1.10) we have

$$|L_f(N;s)| \ge \left|\sum_{n\le N}\frac{a(n)}{n^s}\right| - |\chi_f(s)|\left|\sum_{n\le N}\frac{a(n)}{n^{1-s}}\right|. \tag{5.2}$$

Consider the right-hand side of (5.2). We will obtain an upper bound for the first sum and a lower bound for the second sum. By Lemmas 2.1 and 2.2 we see that

$$\begin{aligned}\left|\sum_{n\le N}\frac{a(n)}{n^s}\right| &\ge |L_f(s)| - \left|\sum_{n>N}\frac{a(n)}{n^s}\right| \\ &> \left(\frac{\sigma-1}{\sigma}\right)^2 - \frac{N^{1-\sigma}}{\sigma-1}\left(\log N + 2\gamma + \frac{1}{\sigma-1}\right) + O\left(\frac{1}{\sqrt{N}}\right). \end{aligned} \tag{5.3}$$

Since by (5.1) we always have

$$\sigma > 1 + \frac{c\log\log N}{\log N},$$

then from (5.3) we have

$$\begin{aligned}\left|\sum_{n\le N}\frac{a(n)}{n^s}\right| &> \left(\frac{c\log\log N}{\log N + c\log\log N}\right)^2 \\ &\quad - \frac{1}{\log^c N}\left(\frac{\log N}{c\log\log N}\right)\left(\log N + 2\gamma + \frac{\log N}{c\log\log N}\right) + O\left(\frac{1}{\sqrt{N}}\right).\end{aligned}$$

Therefore for $c = 4$ one finds that

$$\left|\sum_{n\le N}\frac{a(n)}{n^s}\right| > \left(\frac{\log\log N}{\log N}\right)^2, \tag{5.4}$$

for sufficiently large N. Now by Lemmas 2.2, 2.4, for $|t| > 2\pi e N^{\lambda}$, and $|t| > 20$, we find that

$$|\chi_f(s)|\left|\sum_{n\le N}\frac{a(n)}{n^{1-s}}\right| < 1.02\left(\frac{|s|}{2\pi e}\right)^{1-2\sigma} N^{\sigma}\left(\log N + 2\gamma - 1 + O(N^{-1/2})\right). \tag{5.5}$$

Then from (5.5), for a fixed $\epsilon > 0$ and large N we may write

$$|\chi_f(s)|\left|\sum_{n\le N}\frac{a(n)}{n^{1-s}}\right| < 2.04\left(\frac{|s|}{2\pi e}\right)^{1-2\sigma} N^{\sigma+\epsilon} < 2.04 N^{\lambda(1-2\sigma)+\sigma+\epsilon}. \tag{5.6}$$

If $1/2 < \lambda < 1+\epsilon$, then by (5.1) the exponent of N in (5.6) can be written as

$$\lambda(1-2\sigma)+\sigma+\epsilon = \lambda+\epsilon-\sigma(2\lambda-1) < -c(\lambda+\epsilon)\frac{\log\log N}{\log N} < -c(1+2\epsilon)\frac{\log\log N}{2\log N}.$$

If $\lambda \ge 1+\epsilon$, then the exponent of N in (5.6) is

$$\lambda(1-2\sigma)+\sigma+\epsilon \le (1+\epsilon)(1-2\sigma)+\sigma+\epsilon = (1-\sigma)(1+2\epsilon) < -c(1+2\epsilon)\frac{\log\log N}{\log N}.$$

By combining the above two cases and using (5.6), we derive

$$\left|\chi_f(s)\sum_{n\le N}\frac{a(n)}{n^{1-s}}\right| < \frac{2.04}{\log^{c/2} N}. \tag{5.7}$$

Finally choose $c = 4$. Then from (5.4) and (5.7) we have

$$|L_f(N;s)| > \left(\frac{\log\log N}{\log N}\right)^2 - \frac{2.04}{\log^2 N} > 0, \tag{5.8}$$

for N large enough. Therefore there exists a $N_0 > 0$ such that when $N > N_0$, then $L_f(N;s) \neq 0$ in the region

$$\sigma > \max\left(1, \frac{\lambda+\epsilon}{2\lambda-1}\right)\left(1 + \frac{4\log\log N}{\log N}\right), \quad |t| \geq 2\pi e N^{\lambda},$$

for $\lambda > 1/2$ and any number $\epsilon > 0$. Which completes the proof of Theorem 1.4.

We now prove Theorem 1.5. It is enough to consider the case $N \geq 2$. Suppose $T > T_0$ for some large constant T_0. Let $\sigma \geq 2$ and $|t| > \max(2\pi e N, T_0)$. From (5.2) and using the trivial bound $d(n) \leq n$ we have

$$\begin{aligned} |L_f(N;s)| &\geq |L_f(s)| - \sum_{n>N} \frac{d(n)}{n^{\sigma}} - |\chi_f(s)| \sum_{n \leq N} \frac{d(n)}{n^{1-\sigma}} \\ &> \left(\frac{\sigma-1}{\sigma}\right)^2 - \frac{N^{2-\sigma}}{\sigma-2} - 1.02\left(\frac{|s|}{2\pi e}\right)^{1-2\sigma}\left(N^{\sigma} + \frac{N^{1+\sigma}}{1+\sigma}\right) \\ &> \left(\frac{\sigma-1}{\sigma}\right)^2 - \frac{2^{2-\sigma}}{\sigma-2} - 1.02(2)^{1-2\sigma}\left(2^{\sigma} + \frac{2^{1+\sigma}}{1+\sigma}\right), \end{aligned} \tag{5.9}$$

where in the penultimate step we used Lemma 2.4. We assume in what follows that $T_0 > 20$. A numerical computation shows that the right-hand side of (5.9) is positive when $\sigma \geq 3.5$. Thus $L_f(N;s) \neq 0$ for $\sigma \geq 3.5$ and $|t| > \max(2\pi e N, T_0)$. Also by the functional equation we see that $L_f(N;s) \neq 0$ when $\sigma \leq -2.5$, which concludes the proof of the theorem.

6. Proof of Theorems 1.6 and 1.7

Let $T > 0$ be a large number. Then by Theorem 1.4, we conclude that the zeros of $L_f(N;s)$ with ordinates $T < \gamma_N < T + U$, for some positive constant U, must lie in a rectangle with width $2d - 1$, where $d = \max(1, \lambda/(2\lambda - 1))$. The following theorems will are the main ingredients in the proof of Theorem 1.7.

Theorem 6.1. *Let $\lambda > 1/2$. There exists a constant N_0 such that for $N > N_0$, $T > 2\pi e N^{\lambda}$, and $U \geq 2$, we have*

$$N(T+U) - N(T) = \frac{T+U}{\pi}\log\frac{T+U}{2\pi} - \frac{T}{\pi}\log\frac{T}{2\pi} - \frac{U}{\pi} + O_f\left(\left(\frac{\lambda}{2\lambda-1}\right)^3 \log(T+U)\right). \tag{6.1}$$

Furthermore there exists a constant T_0 such that (6.1) holds with $\lambda = 1$ for all $N \geq 1$ and $T > \max(2\pi e N, T_0)$.

Proof. Let $\lambda > 1/2$ and $w = \max\left(3, \frac{2\lambda}{2\lambda-1}\right)$. Let R be a positively oriented rectangle with vertices $w + iT$, $w + i(T+U)$, $1 - w + i(T+U)$ and $1 - w + iT$. From Theorem 1.4, we observe that the complex zeros will be inside the rectangle R for sufficiently large N. Without loss of generality we assume that the edges of the rectangle do not pass through any zeros of

$L_f(N;s)$. Then by Littlewood's lemma [20, Section 9.9] we have

$$2\pi\sum_{\rho\in R}(\beta_N-1+w)=\int_T^{T+U}(\log|L_f(N;1-w+it)|-\log|L_f(N;w+it)|)\,dt$$
$$+\int_{1-w}^{w}(\arg L_f(N;\sigma+i(T+U))d-\arg L_f(N;\sigma+iT))\,d\sigma, \tag{6.2}$$

where the argument of $L_f(N;s)$ is obtained by continuation of $\log L_f(N;s)$ leftward from the value 0 at $\sigma=\infty$. From (1.10) we have

$$L_f(N;s)=1+\sum_{2\le n\le N}\frac{a(n)}{n^s}+\chi_f(s)\sum_{1\le n\le N}\frac{a(n)}{n^{1-s}}.$$

Then from (1.2) we may write

$$|L_f(N;s)-1|\le\sum_{2\le n\le N}\frac{d(n)}{n^\sigma}+|\chi_f(s)|\sum_{1\le n\le N}\frac{|a(n)|}{n^{1-\sigma}}.$$

Since $T\ge 2\pi eN^\lambda$, applying (1.4) and (5.6) we find that

$$\begin{aligned}|L_f(N;s)-1|&\le\sum_{2\le n\le N}\frac{1}{n^{\sigma-1}}+O\big(N^{\lambda(1-2\sigma)+\sigma+\epsilon}\big)\\&\le\frac{1}{2^{\sigma-1}}+\int_2^N\frac{1}{x^{\sigma-1}}\,dx+O\big(N^{\lambda(1-2\sigma)+\sigma+\epsilon}\big)\\&\le\frac{\sigma}{(\sigma-2)2^{\sigma-1}}+O\big(N^{\lambda(1-2\sigma)+\sigma+\epsilon}\big)\\&<\frac{4}{5},\end{aligned}\tag{6.3}$$

for $\sigma\ge w$ and large N. Therefore from (6.3), $\log L_f(N;s)$ is analytic and non-zero for $\sigma\ge w$. Then by Cauchy's theorem,

$$\int_T^{T+U}\log L_f(N;w+it)\,dt=\int_w^\infty\log L_f(N;\sigma+iT)\,d\sigma-\int_w^\infty\log L_f(N;\sigma+i(T+U))\,d\sigma. \tag{6.4}$$

Again from (6.3), the integrals on the right-hand side of (6.4) are bounded. Therefore

$$-\int_T^{T+U}\log|L_f(N;w+it)|dt=-\operatorname{Re}\int_T^{T+U}\log L_f(N;w+it)\,dt=O(1). \tag{6.5}$$

Using the functional equation (1.11) we may write

$$\int_T^{T+U}\log|L_f(N;1-w+it)|\,dt=\int_T^{T+U}\log|L_f(N;w+it)|\,dt-\int_T^{T+U}\log|\chi_f(w+it)|\,dt. \tag{6.6}$$

Recall Stirling's formula in the form

$$\log\Gamma(s)=\left(s-\frac12\right)\log s-s+\frac12\log 2\pi+O\left(\frac{1}{|s|}\right), \tag{6.7}$$

as $|s| \to \infty$ and $|\arg s| \le \pi - \epsilon$. Then from (1.6) and (6.7) we have

$$\begin{aligned}\log \chi_f(s) &= (2s-1)\log 2\pi + \log \Gamma\left(\frac{k+1}{2} - s\right) - \log \Gamma\left(\frac{k-1}{2} + s\right)\\ &= (2s-1)\log 2\pi + \left(\frac{k}{2} - s\right)\left(\log s - i\pi - \frac{k+1}{2s} + O\left(\frac{1}{|s|^2}\right)\right) - \left(\frac{k+1}{2} - s\right)\\ &\quad - \left(\frac{k}{2} - 1 + s\right)\left(\log s + \frac{k-1}{2s} + O\left(\frac{1}{|s|^2}\right)\right) + \left(\frac{k-1}{2} + s\right) + O\left(\frac{1}{|s|}\right)\\ &= (1-2s)\log\frac{s}{2\pi} - \frac{i\pi}{2}(k-2s) + 2s + O_k\left(\frac{1}{|s|}\right) \qquad (6.8)\end{aligned}$$

Note that

$$\int_T^{T+U} \log|\chi_f(w+it)|\,dt = \text{ Re } \int_T^{T+U} \log\chi_f(w+it)\,dt = \text{ Im } \int_{w+iT}^{w+i(T+U)} \log\chi_f(s)\,ds. \qquad (6.9)$$

Also for $t \to \infty$

$$\text{Re } (\log s) = \log t + O\left(\frac{\sigma^2}{t^2}\right) \quad \text{and} \quad \text{Im } (\log s) = \left(\frac{\pi}{2} - \frac{\sigma}{t}\right) + O\left(\frac{\sigma^3}{t^3}\right). \qquad (6.10)$$

Therefore from (6.8), (6.9), (6.10), a straightforward computation shows that

$$\begin{aligned}\int_T^{T+U} \log|\chi_f(w+it)|\,dt &= (1-2w)(T+U)\log\frac{T+U}{2\pi} - (1-2w)T\log\frac{T}{2\pi}\\ &\quad - (1-2w)U + O_f(w^3\log(T+U)). \qquad (6.11)\end{aligned}$$

Hence from (6.5), (6.6) and (6.11) we find that

$$\begin{aligned}\int_T^{T+U} \log|L_f(N;1-w+it)|\,dt &= (2w-1)(T+U)\log\frac{T+U}{2\pi} - (2w-1)T\log\frac{T}{2\pi}\\ &\quad - (2w-1)U + O_f(w^3\log(T+U)). \qquad (6.12)\end{aligned}$$

Next we consider the change in $\arg L_f(N;s)$ along the bottom edge of R. Let q be the number of zeros of Re $(L_f(N;\sigma+iT))$ on the interval $(1-w,w)$. Then there are at most $q+1$ subintervals of $(1-w,w)$ in each of which Re $(L_f(N;\sigma+iT))$ is of constant sign. Therefore the variation of $\arg L_f(N;\sigma+iT)$ is at most π in each subinterval. So we have

$$\arg L_f(N;\sigma+iT)\big|_{1-w}^{w} \le (q+1)\pi. \qquad (6.13)$$

To estimate q, first we define

$$g(z) := L_f(N;z+iT) + \overline{L_f(N;\bar{z}+iT)}. \qquad (6.14)$$

If $z=\sigma$ is a real number then we have

$$g(\sigma) = \text{ Re } (L_f(N;\sigma+iT)). \qquad (6.15)$$

Let $R = 2(2w-1)$ and consider the disk $|z-w| < R$ centered at w. Choose T large so that

$$\text{Im } (z+iT) > T - R > 0.$$

Thus, $L_f(N;z+iT)$, and hence also $g(z)$, are analytic in the disk $|z-w| < R$. Let $n(r)$ be the number of zeros of $g(z)$ in the disk $|z-w| < r$ and $R_1 = R/2$. Then we have

$$\int_0^R \frac{n(r)}{r}\,dr \ge n(R_1)\int_{R_1}^R \frac{dr}{r} = n(R_1)\log 2. \qquad (6.16)$$

By Jensen's theorem,

$$\int_0^R \frac{n(r)}{r}\,dr = \frac{1}{2\pi}\int_0^{2\pi} \log\frac{|g(w+Re^{i\theta})|}{|g(w)|}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log|g(w+Re^{i\theta})|\,d\theta - \log|g(w)|. \tag{6.17}$$

From (6.3) we have

$$|\text{ Re }(L_f(N;w+iT))| > \frac{1}{5}$$

and hence from (6.15) we find

$$|g(w)| > \frac{1}{5}.$$

From the definition (1.10) we have

$$|L_f(N;s)| \le \sum_{n\le N}\frac{d(n)}{n^\sigma} + |\chi_f(s)|\sum_{n\le N}\frac{d(n)}{n^{1-\sigma}}.$$

By Lemma 2.4, we have

$$\chi_f(s) \ll |s|^{(1-2\sigma)}.$$

One can show (similar to Lemma 2.2) that

$$\sum_{n\le N}\frac{d(n)}{n^\sigma} \ll \begin{cases} N^{1-\sigma}\log N & \text{if } \sigma\ne 1\\ \log^2 N & \text{if } \sigma = 1\end{cases}.$$

Thus,

$$|L_f(N;s+iT)| \ll \log N(N^{1-\sigma}+\log N + T^{1-2\sigma}N^\sigma).$$

Therefore from (6.14), we have

$$|g(s)| \le |L_f(N;s+iT)| + |L_f(N;s-iT)| \ll \log N(N^{1-\sigma}+\log N+T^{1-2\sigma}N^\sigma). \tag{6.18}$$

Since $|s-w| < R = 2(2w-1)$, then $2-3w<\sigma<5w-2$. Also $T\ge 2\pi eN^\lambda$ for $\lambda > 1/2$. So the expression on the right-hand side of (6.18) is largest when $\sigma = 3-2w$. Therefore

$$\begin{aligned} |g(s)| &\ll \log N(N^{3w-1}+\log N + TN^{(2\lambda-1)(3w-2)}\\ &\ll \log T(T^{(3w-1)/\lambda} + T^{1+(2\lambda-1)(3w-2)/\lambda})\\ &\ll T^{6w}. \end{aligned} \tag{6.19}$$

Finally

$$|g(w+Re^{i\theta})| \ll T^{6w}.$$

Hence from (6.16) and (6.17), it follows that $n(R_1) \ll w\log T$. Now, the zeros of $L_f(N;\sigma+iT)$ for $1-w<\sigma<w$ correspond to, and their number equals the number of, the zeros of $g(\sigma)$ in the same interval. Since the interval $(1-w,w)$ is contained in the disk $|s-w|<R_1$, then $q\le n(R_1)$. Since

$$w = \max\left(3, \frac{2\lambda}{2\lambda-1}\right) \le \frac{6\lambda}{2\lambda-1},$$

then from (6.13) we conclude that

$$\int_{1-w}^{w} \arg L_f(N;\sigma+iT)\,d\sigma \ll \left(\frac{\lambda}{2\lambda-1}\right)^3\log T. \tag{6.20}$$

Similarly,

$$\int_{1-w}^{w} \arg L_f(N;\sigma+i(T+U))\,d\sigma \ll \left(\frac{\lambda}{2\lambda-1}\right)^3 \log(T+U). \tag{6.21}$$

For smaller values of N one can obtain similar results as (6.3) to (6.21) by choosing the rectangular contour $R=[3.5+iT, 3.5+i(T+U), -2.5+i(T+U), -2.5+iT]$ and $T>\max(2\pi eN, T_0)$. Here T_0 is the same as in Theorem 1.5. Combining (6.2), (6.5), (6.12), (6.20), and (6.21), we have the following result.

Theorem 6.2. *For $\lambda > 1/2$, $N \geq N_0$, and $T \geq 2\pi e N^{\lambda}$, we have*

$$2\pi\sum_{\rho\in R}(\beta_N-1+w) = (2w-1)(T+U)\log\frac{T+U}{2\pi} - (2w-1)T\log\frac{T}{2\pi} - (2w-1)U + O_f\left(\left(\frac{\lambda}{2\lambda-1}\right)^3\log(T+U)\right). \tag{6.22}$$

Furthermore there exists a constant T_0 such that (6.22) holds with $\lambda=1$ for all $N\geq 1$ and $T>\max(2\pi eN, T_0)$.

Now increasing w to $w+1$ in Theorem 6.2 and subtracting (6.22) from the corresponding relation where w is replaced by $w+1$ gives the conclusion of Theorem 6.1.

□

Theorem 6.3. *There exists a constant T_0 such that if $N\geq 1$, $T>\max(2\pi eN, T_0)$, and $U\geq 2$, then*

$$N^0(T+U)-N^0(T) \geq \frac{T+U}{\pi}\log\frac{T+U}{2\pi M^a} - \frac{T}{\pi}\log\frac{T}{2\pi M^a} - \frac{U}{\pi} + O_f(N), \tag{6.23}$$

where $0\leq a\leq 1$ is such that the number of zeros of $\sum_{n\leq N} a(n)n^{-s}$ with real parts strictly greater than $1/2$ is

$$\leq \frac{aT}{2\pi}\log M + O_f(N).$$

Also, the right-hand side of (6.23) is a lower bound for the number of distinct zeros of $L_f(N;s)$ on the critical line with $T<t\leq T+U$. Here M is defined in Proposition 1.1.

Proof. First of all we introduce some notation to simplify the proof. Rewrite (1.10) in the form

$$L_f(N;s) = F(s)\left(1+\chi_f(s)\frac{F(1-s)}{F(s)}\right) = F(s)Z(s), \tag{6.24}$$

where

$$F(s) := \sum_{n\leq N}\frac{a(n)}{n^s}$$

and

$$Z(s) = 1+\chi_f(s)\frac{F(1-s)}{F(s)}.$$

Define

$$\begin{aligned}
N_F(T) &= \#\{\rho : F(\rho) = 0 \text{ and } 0 < \text{ Im } \rho \le T\},\\
N_Z(T) &= \#\{\rho : Z(\rho) = 0 \text{ and } 0 < \text{ Im } \rho \le T\},\\
N_F^0(T) &= \#\{\rho : F(\rho) = 0, \text{ Re } \rho = 1/2 \text{ and } 0 < \text{ Im } \rho \le T\},\\
N_Z^0(T) &= \#\{\rho : Z(\rho) = 0, \text{ Re } \rho = 1/2 \text{ and } 0 < \text{ Im } \rho \le T\},\\
N_F^+(T) &= \#\{\rho : F(\rho) = 0, \text{ Re } \rho > 1/2 \text{ and } 0 < \text{ Im } \rho \le T\},
\end{aligned}$$

and

$$N_Z^+(T) = \#\{\rho : Z(\rho) = 0, \text{ Re } \rho > 1/2 \text{ and } 0 < \text{ Im } \rho \le T\}.$$

Clearly $N(X;T) = N_F(T)$ for $X = N$. Also $N^0(T) = N_F^0(T) + N_Z^0(T)$. From (6.24) we see that $L_f(N; \frac{1}{2} + it) = 0$ if and only if $F(\frac{1}{2} + it) = 0$ or $Z(\frac{1}{2} + it) = 0$. If $1/2 + ig$ is a zero of $F(s)$ then we write

$$Z(1/2 + ig) = 1 + \chi_f(1/2 + ig) \lim_{t \to g} \frac{F(1/2 - it)}{F(1/2 + it)}.$$

Our next goal is to provide a lower bound for $N_Z^0(T + u) - N_Z^0(T)$, or equivalently, obtain a lower bound for the umber of solutions of

$$\chi_f(1/2 + it)\frac{F(1/2 - it)}{F(1/2 + it)} = -1,$$

for $T \le t \le T + U$. Note that if

$$\chi_f(1/2 + it)\frac{F(1/2 - it)}{F(1/2 + it)} = -1,$$

then

$$\arg\left(\chi_f(1/2 + it)\frac{F(1/2 - it)}{F(1/2 + it)}\right) = (2m + 1)\pi$$

and hence

$$\arg \chi_f(1/2 + it) - 2 \arg F(1/2 + it) = (2m + 1)\pi$$

for some integer m. Let

$$G(s) := \arg \chi_f(s) - 2 \arg F(s).$$

Fix $\epsilon > 0$. Construct a continuous curve $\mathcal{L}(\epsilon)$ from $1/2 + iT$ to $1/2 + i(T + U)$ directed upward, which is the union of line segments belonging to the same vertical line and any two consecutive segments joint by a small semicircle of radius ϵ as follows. The semi circles have the same radius $\epsilon > 0$, are centered exactly at the zeros $1/2 + ig$ of $F(s)$, and lie to the right of the critical line. Here we chose ϵ small enough so that the semicircles do not overlap. Next consider a straight line segment of $\mathcal{L}(\epsilon)$ between two consecutive zeros of $F(s)$, excluding the semicircle part. Each time the image under $G(s)$ of this straight line segment crosses the horizontal lines $y = (2m + 1)\pi$ for $m \in \mathbb{Z}$, it gives rise to a distinct zero of $Z(1/2 + it)$. Furthermore, by the argument principle, as $\epsilon \to 0^+$ the image of the small semicircle under $G(s)$ is a vertical line segment of length $\pi m(g)$, where $m(g)$ is the multiplicity of the zero $1/2+ig$ of $F(s)$. In the limit, the function $G(s)$ has a jump discontinuity at each zero $1/2+ig$ of $F(s)$ with jump $\pi m(g)$.

Consider a rectangle of height H with horizontal grid lines, such that the distance between any two consecutive lines is equal to 2π. If a continuous curve intersects all the horizontal grid lines then the minimum number of points of intersection is $H/2\pi$. Using this geometrical fact, we see that the number of zeros of $Z(s)$ arising from the image of the straight line segment of $\mathcal{L}(\epsilon)$ crossing the lines $y = (2m+1)\pi$ is at least

$$\lim_{\epsilon\to 0^+} \frac{1}{2\pi} |\Delta_{\mathcal{L}(\epsilon)}(\arg \chi_f(s) - 2\arg F(s)| + O(1).$$

In particular, if J is the total number of crossings of the set of jumps by the lines $y = (2m+1)\pi$ then

$$\lim_{\epsilon\to 0^+} \frac{1}{2\pi} |\Delta_{\mathcal{L}(\epsilon)}(\arg \chi_f(s) - 2\arg F(s)| - J + O(1) \tag{6.25}$$

gives a lower bound for the number of distinct zeros of $Z(1/2+it)$ with $T \leq t \leq T+U$. We take this quantity as a lower bound for $N_Z^0(T+u) - N_Z^0(T)$. Since any vertical line of length $\pi m(g)$ crosses the lines $y = (2m+1)\pi$ at most $m(g)$ times then we have

$$J \leq \sum_{T\leq g\leq T+U} m(g).$$

Hence

$$J \leq N_F^0(T+U) - N_F^0(T). \tag{6.26}$$

To estimate $\triangle_{\mathcal{L}(\epsilon)} \arg F(s)$, we will consider a clockwise oriented contour $C(\epsilon)$ from by $\mathcal{L}(\epsilon)$ and the line segments $(\frac{1}{2}+i(T+U), 3.5+i(T+U)]$, $[3.5+iT, 3.5+i(T+U)]$, and $(\frac{1}{2}+i(T+U), 3.5+iT]$. We have

$$\Delta_{C(\epsilon)} \arg F(s) = -2\pi(N_F^+(T+U) - N_F^+(T)).$$

From the definition of $F(s)$ and an argument similar to (6.3) we find

$$|F(s) - 1| < 1.$$

Hence

$$\arg F(3.5+it)|_T^{T+U} = O(1).$$

Note that

$$\text{Im }(F(\sigma+iT)) = -\sum_{n\leq N} \frac{a(n)\sin(T\log n)}{n^\sigma}.$$

By a generalization of Descartes's Rule of Signs (see Pólya and Szegö [16], Part V, Chapter 1, No. 77), the number of real zeros of $\text{Im }(F(\sigma+iT))$ in the interval $1/2 \leq \sigma \leq 3.5$ is less than or equal to the number of sign changes in the sequence $a(n)\sin(T\log n)$, $1 \leq n \leq N$, which in turn is less than or equal to the number of nonzero coefficients of $a(n)\sin(T\log n)$. Therefore

$$\arg F(\sigma+iT)|_{1/2}^w = O_f(N).$$

Similarly

$$\arg F(\sigma+i(T+U))|_{1/2}^w = O_f(N).$$

Thus

$$\triangle_{\mathcal{L}(\epsilon)} \arg F(s) = -2\pi(N_F^+(T+U) - N_F^+(T)) + O_f(N). \tag{6.27}$$

Again by (6.8),

$$\begin{aligned}\Delta_{\mathcal{L}(\epsilon)} \arg \chi_f(s) &= -\arg \chi_f(1/2+it)|_T^{T+U} + O_f(1) \\ &= -2(T+U)\log\frac{T+U}{2\pi} + 2T\log\frac{T}{2\pi} + 2U + O_f(1). \end{aligned} \tag{6.28}$$

Finally combining (6.25), (6.26), (6.27), and (6.28) we obtain

$$\begin{aligned} N_Z^0(T+u) - N_Z^0(T) \geq \frac{T+U}{\pi}\log\frac{T+U}{2\pi} - \frac{T}{\pi}\log\frac{T}{2\pi} - \frac{U}{\pi} - 2(N_F^+(T+U) - N_F^+(T)) \\ - (N_F^0(T+U) - N_F^0(T)) + O_f(N). \end{aligned}$$

Now by Proposition 1.1 there exists a positive number a with $0 \leq a \leq 1$ such that

$$N_F^+(T+U) - N_F^+(T) \leq a\frac{U}{2\pi}\log M + O_f(N).$$

Thus

$$\begin{aligned} N^0(T+U) - N^0(T) &= N_Z^0(T+u) - N_Z^0(T) + N_F^0(T+U) - N_F^0(T) \\ &\geq \frac{T+U}{\pi}\log\frac{T+U}{2\pi} - \frac{T}{\pi}\log\frac{T}{2\pi} - \frac{U}{\pi} - \frac{aU}{\pi}\log M + O_f(N), \end{aligned} \tag{6.29}$$

which proves Theorem 6.3.

□

For $\lambda > 1/2$, one derives from (6.1) that

$$\begin{aligned} N^0(T+U) - N^0(T) &\geq \frac{T+U}{\pi}\log\frac{T+U}{2\pi e M^a} - \frac{T}{\pi}\log\frac{T}{2\pi e M^a} - \frac{U}{\pi} + O_f(N) \\ &= \frac{T+U}{\pi}\log\frac{T+U}{2\pi} - \frac{T}{\pi}\log\frac{T}{2\pi} - \frac{U}{\pi} + O_f(U\log N) + O_f(N) \\ &= N(T+U) - N(T) + O(U\log N) + O_f(N) + O_f\left(\left(\frac{\lambda}{2\lambda-1}\right)^3 \log(T+U)\right), \end{aligned} \tag{6.30}$$

which completes the proof of Theorem 1.6. Now for $N \leq T^{o(1)}$ and For $U \geq T^{\beta}$ for some positive constant β, we have

$$\liminf_{T\to\infty} \frac{N^0(T+U) - N^0(T)}{N(T+U) - N(T)} = 1. \tag{6.31}$$

Since the right-hand sides of (6.29) and (6.30) are also lower bounds for the number of simple zeros of $L_f(N; 1/2+it)$ with $T \leq t \leq T+U$, then the lim inf in (6.31) continues to equal 1 when one replaces $N^0(T+U) - N^0(T)$ on the left-hand side of (6.31) by the number of simple zeros of $L_f(N; 1/2+it)$ with $T \leq t \leq T+U$. This implies that as $T \to \infty$, 100% of the zeros of $L_f(N; s)$ are simple and lie on the critical line, which concludes the proof of Theorem 1.7.

Acknowledgement

The authors are grateful to the referee for many useful comments and suggestions.

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