VALUE DISTRIBUTION OF $L'(\rho)$

JUNXIAN LI AND ALEXANDRU ZAHARESCU

Abstract. Let L be an automorphic L-function. Assuming the Riemann Hypothesis for $L(s)$ and the Selberg normality conjecture, we obtain a lower bound for the second negative moment and extreme small values of $L'(\rho)$, where ρ is a zero of $L(s)$.

1. Introduction

We first introduce a class $\mathcal{S}$ which consists of L-functions with the following properties.

(1) Dirichlet series representation: For $\Re(s) > 1$, $L(s)$ can be represented as an absolutely convergent Dirichlet series $L(s) = \sum_n \frac{a(n)}{n^s}$.

(2) Analytic continuation: There exists a non-negative integer m such that

$$(s-1)^m L(s) \tag{1}$$

is an entire function of finite order.

(3) Functional equation: $L(s)$ satisfies the functional equation

$$\Xi_L(s) = w_L \overline{\Xi_L(1-\bar{s})} =: \omega_L \Xi_{\bar{L}}(1-s),$$

where

$$\Xi_L(s) := L(s)Q^s \prod_{j=1}^{f} \Gamma(\lambda_j s + \mu_j) =: L(s)Q^s\gamma_L(s),\ \bar{L}(s) = \overline{L(\bar{s})}, \tag{2}$$

and the parameters $f \geq 0, Q > 0, \lambda_j > 0$ are real numbers and μ_j, w_L are complex numbers satisfying $\Re\mu_j \geq 0, |w_L| = 1$.

(4) Euler product: For $\Re(s)$ sufficiently large, $L(s)$ has the Euler product representation

$$L(s) = \prod_p L_p(s),\ L_p(s) = \exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right), \tag{3}$$

where $b(p^k)$ are some coefficients satisfying $b(p^k) \ll p^{k\theta_L}$, for some constant $\theta_L < 1/2$.

2010 *Mathematics Subject Classification.* 11M06, 11R42.
Key words and phrases. Selberg class, derivative, Dedekind zeta function, extreme values.

(5) The degree of $L(s)$ is defined as $d_L = 2\sum_{j=1}^{f}\lambda_j$ and the arithmetic conductor of $L(s)$ is defined as $q_L = (2\pi)^{d_L}Q^2\prod_{j=1}^{f}\lambda_j^{2\lambda_j}$. Define the analytic conductor as

$$C_L(s) = q_L\prod_{j=1}^{f}(|s+\mu_j|+3)^{2\lambda_j}, \tag{4}$$

where μ_j and Q are defined in (2).

If one further assumes the Ramanujan conjecture, which says that $a_n \ll_\epsilon n^\epsilon$ for any fixed $\epsilon > 0$, then this class of L-functions is known as the Selberg class. The Riemann zeta function, Dirichlet L-functions, the Dedekind zeta function of a number field, and L-functions associated to holomorphic cusp forms are all examples of functions in the Selberg class. However, there are also many examples of L-functions where the Ramanujan conjecture is not known. Thus the above class $\mathcal{S}$ contains a larger class of L-functions, such as automorphic L-functions of $GL(m)$. We are interested in studying the value distribution of $L'(\rho)$ for a given $L\in\mathcal{S}$. We establish a lower bound for the negative moment of $L'(\rho)$ for $L\in\mathcal{S}$ under the stronger form of Selberg's normality conjecture.

Theorem 1.1 *Assume $L\in\mathcal{S}$ and L satisfies the Selberg normality conjecture*

$$\sum_{p\le x}\frac{|a(p)|^2\log p}{p} = \kappa\log x + O(1). \tag{5}$$

If $L(s)$ has no zeros on $\Re(s) > \frac{1}{2}$, then

$$\sum_{T\le\Im\rho\le 2T}\frac{1}{|L'(\rho)|^2} \gg T(\log T)^{\kappa-1},$$

where the implied constant depends on L and can be computed explicitly.

In the case of $L = \zeta(s)$, this is a result of Gonek [2], The constant has been made explicit by Milinovich and Ng [7]. Theorem (1.1) shows that $L'(\rho)$ can be as small as $(\log|\Im\rho|)^{-\kappa+1}$. In fact, one can prove a stronger result.

Theorem 1.2 *Assume $L\in\mathcal{S}$ and L satisfies the Selberg normality conjecture*

$$\sum_{p\le x}|a(p)|^2 = (\kappa+o(1))\frac{x}{\log x}. \tag{6}$$

If $L(s)$ has no zeros on $\Re(s) > \frac{1}{2}$, then there are infinitely many zeros ρ of $L(s)$ such that

$$\min_{T\le\Im\rho\le 2T}|L'(\rho)| \ll \exp\left(-(\sqrt{\kappa}+o(1))\frac{\log T}{\log\log T}\right).$$

If $L = \zeta_K(s)$, where $K/\mathbb{Q}$ is a Galois extension of degree n_0, then from [8, Lemma 5.2], we have

$$\sum_{p\le x}|a(p)|^2 = (n_0+o(1))\frac{x}{\log x}.$$

Thus, as a corollary of Theorem 1.2 we have

Corollary 1.3 *Let $K/\mathbb{Q}$ be a Galois extension of degree n_0 and let $\zeta_K(s)$ be the Dedekind zeta function of K. If all nontrivial zeros of $\zeta_K(s)$ are on the line $\Re(s) = \frac{1}{2}$, then*

$$\min_{T\leq\Im\rho\leq 2T} |\zeta_K'(\rho)| \ll \exp\left(-\sqrt{\frac{n_0\log T}{\log\log T}}\right),$$

$$\max_{T\leq\Im\rho\leq 2T} \left|\operatorname{Res}\zeta_K^{-1}(s)\big|_{s=\rho}\right| \gg \exp\left(\sqrt{\frac{n_0\log T}{\log\log T}}\right),$$

where $\rho = \frac{1}{2} + i\gamma$ is a zero of $\zeta_K(s)$ and c is some positive constant.

If K is an abelian extension of $\mathbb{Q}$, then all zeros of $\zeta_K(s)$ are conjectured to be simple, in which case $\zeta_K'(\rho)$ cannot be zero. If K is a cyclotomic field $K = \mathbb{Q}(\zeta_q)$, then $\zeta_K(s) = \prod_\chi L(s,\chi)$, where χ runs through all Dirichlet characters modulo q. The conjecture on simplicity of the zeros of $\zeta_K(s)$ is a consequence of the Linear Independence conjecture (LI), or the Grand Simplicity Hypothesis (GSH), which says that non-negative imaginary parts of the non-trivial zeros of Dirichlet L-functions corresponding to primitive characters are linearly independent over the rationals (see Wintner [15], Hooley [3], Montgomery [9], Rubinstein and Sarnak [11]). If $\zeta_K'(\rho) \neq 0$, it is natural to ask how small $|\zeta_K'(\rho)|$ can be. When $K = \mathbb{Q}$, Corollary (1.3) recovers a result of Ng [10] on small values of $|\zeta'(\rho)|$.

The conditions (5) and (6) are related to Selberg's orthonormality conjecture.

Conjecture 1.4 (Selbergs orthonormality conjecture) *Let L be in the Selberg class. Then there exits some constant κ depending on L such that*

$$\sum_{p\leq x} \frac{|a(p)|^2}{p} = \kappa \log\log x + O(1). \tag{7}$$

For distinct primitive functions L_1, L_2 in the Selberg class,

$$\sum_{p\leq x} \frac{a_{L_1}(p)\overline{a_{L_2}(p)}}{p} = O(1). \tag{8}$$

Here $F \in \mathcal{S}\backslash\{1\}$ is said to be primitive if $F = F_1F_2$ with F_1, $F_2 \in \mathcal{S}$ implies $F_1 = 1$ or $F_2 = 1$.

There are examples for which the Selberg normality conjecture is known. Let π be an irreducible automorphic cuspidal representation of $GL(m,\mathbb{A})$. Then for $m \leq 4$, (7) holds true. This is clear when $m = 1$, and when $m = 2$ it follows from known bounds towards the Ramanujan conjecture [12]. For $m = 3$, it was proved by Rudnick and Sarnak [12], and for $m = 4$, it was proved by Kim and Sarnak [5]. Liu and Ye [6] have obtained further results related to (8).

2. Overview of the proof

We follow the approaches in [7] and [10], which involve asymptotic formulas for mollified moments of $L'(\rho)$. Let $X(s) = \sum_{n\leq M} x_n n^{-s}$, and $Y(s) = \sum_{n\leq M} y_n n^{-s}$ be

Dirichlet polynomials. Consider

$$S_0 = \sum_{\substack{L(\rho)=0 \\ T_1<\Im\rho<T_2}} X(\rho)Y(1-\rho), \tag{9}$$

$$S_1 = \sum_{\substack{L(\rho)=0 \\ T_1<\Im\rho<T_2}} L'(\rho)^{-1}X(\rho)Y(1-\rho), \tag{10}$$

$$S_2 = \sum_{T_1\le\Im\rho\le T_2} \frac{1}{L'(\rho)}\overline{X}(1-\rho). \tag{11}$$

where $T_1 = T + O(1)$ and $T_2 = 2T + O(1)$ are chosen such that they are $\gg \frac{1}{\log T}$ away from ordinates of zeros of $L(s)$. Then, we further adjust T_1 and T_2 such that $T_1 = T + O(1)$, $T_2 = 2T + O(1)$ and $L(\sigma + iT_i) \gg T_i^{-\epsilon}$. This is possible by Proposition (3.1). If $Y(s) = \overline{X}(s)$, then we have

$$X(\rho)Y(1-\rho) = |X(\rho)|^2$$

since we assume that $\Re(\rho) = \frac{1}{2}$. We have

$$\sum_{T_1\le\Im\rho\le T_2} \frac{1}{|L'(\rho)|^2} \ge \frac{|S_2|^2}{S_0}, \tag{12}$$

and

$$\min_{\substack{L(\rho)=0 \\ T_1<\Im\rho<T_2}} |L'(\rho)| \le \frac{S_0}{|S_1|}. \tag{13}$$

Then, Theorem 1.1 and Theorem 1.2 follow by certain choices of x_n, y_n. For Theorem 1.1 we chose x_n to mimic $L(s)^{-1}$, and for Theorem 1.2 we chose x_n to be the "resonator" coefficients, introduced by Soundararajan [13] to study extreme values of $\zeta(s)$ and other L-functions.

The paper is organized as follows. In Section 3 we list some key propositions and lemmas, among which one of them is proved in Section 7. In Section 4, we provide asymptotic formulae for S_1 and S_0 in Theorem 4.1 and Theorem 4.2 respectively. The formula for S_2 can be derived from S_1. In Section 5 and Section 6, we present the proof of Theorem 1.1 and Theorem 1.2 respectively.

3. Preliminaries

Proposition 3.1 *Let $L \in \mathcal{S}$. Each interval $[T, T+1]$ contains a value of t such that*

$$|L(\sigma+it)| \ge \exp\left(-A\frac{\log t}{\log\log t}\right), \ \frac{1}{2} \le \sigma \le 2.$$

Proof. The proof follows as in the case of the Riemann zeta function. For completeness, we provide a proof in Section 7. □

Lemma 3.2 *Let $L \in \mathcal{S}$. Denote*

$$L(s)^{-1} := \sum_{n=1}^{\infty} \frac{a^{-1}(n)}{n^s}, \quad \text{for } \Re(s) > 1. \tag{14}$$

Then, for any ϵ, there exists $z = z(\epsilon)$ such that

$$|a^{-1}(n)| \ll n^{\theta_L + \epsilon}$$

for all $(n, z) = 1$, where θ_L is a constant less than $\frac{1}{2}$. Also, for all primes p, we have

$$|a^{-1}(p^k)| \ll e^k p^{k\theta_L}.$$

Proof. From (3), we have

$$L(s)^{-1} = \prod_p L_p(s)^{-1} = \prod_p \exp\left(-\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right),$$

thus

$$a^{-1}(p^k) = \sum_{r_1 + 2r_2 + \cdots + kr_k = k} \frac{(-1)^{r_1 + \cdots r_t} b(p)^{r_1} b(p^2)^{r_2} \cdots b(p^k)^{r_k}}{r_1! \cdots r_k!}.$$

Since $|b(p^k)| \leq p^{k\theta_L}$, we have

$$a^{-1}(p^k) \ll e^k p^{k\theta_L}$$

for all p. For any ϵ, there exists p_z such that $e^k \leq p^{k\epsilon}$ for all $p \geq p_z$. Therefore, for $(n, \prod_{p \leq p_z} p) = 1$, we have $|a^{-1}(n)| \ll n^{\theta_L + \epsilon}$ by multiplicativity. □

Proposition 3.3 *If n is squarefree, then $a^{-1}(n) = \mu(n)a(n)$.*

Proof. We have $L(s)\frac{1}{L}(s) = 1$, $a(n)$ is multiplicative, $a^{-1}(n)$ is multiplicative, $a(1) = 1$ and

$$a^{-1}(p) = -\sum_{d|p, d>1} a(p)a^{-1}(p/d) = -a(p)a^{-1}(1) = -\frac{a(p)}{a(1)} = -a(p),$$

since $a(1)a^{-1}(1) = 1$. □

Lemma 3.4 *Let $L \in \mathcal{S}$. Then,*

$$\sum_{n=1}^{\infty} \frac{|a^{-1}(n)|}{n^2} \ll 1,$$

$$\sum_{n=1}^{\infty} \frac{\Lambda_L(n)}{n^2} \ll 1.$$

Proof. From Lemma 3.2, for any $\epsilon > 0$, there exists z such that $a^{-1}(n) \ll n^{\theta_L + \epsilon}$ for all $(n, z) = 1$. By the multiplicativity of $a^{-1}(n)$, we have

$$\sum_{n=1}^{\infty} \frac{|a^{-1}(n)|}{n^2} = \prod_{p|z} \exp\left(1 + \sum_{k=1}^{\infty} \frac{|a^{-1}(p^k)|}{p^{2k}}\right) \sum_{\substack{n=1 \\ (n,z)=1}}^{\infty} \frac{|a^{-1}(n)|}{n^2}.$$

From Lemma 3.2, we have $|a^{-1}(p^k)| \ll e^k p^{k\theta_L}$. It then follows that

$$1 + \sum_{k=1}^{\infty} \frac{|a^{-1}(p^k)|}{p^{2k}} \ll \frac{p^2}{p^2 - ep^{\theta_L}} \ll 1$$

since $2^{2-\theta_L} > e$. Thus,

$$\sum_{n=1}^{\infty} \frac{|a^{-1}(n)|}{n^2} \ll 1.$$

Since $\lambda_L(p^k) = kb(p^k)\log p$, and $b(p^k) \ll p^{k\theta_L}$, we have

$$\sum_{n=1}^{\infty} \frac{\Lambda_L(n)}{n} \ll \sum_p \sum_k \frac{p^{k\theta_L}\log p}{p^{2k}} \ll \sum_n \frac{1}{n^{3/2-\epsilon}} \ll 1.$$

□

Lemma 3.5 (Convexity Bound) *For any $0 < \sigma < 1$ and any $\epsilon > 0$, there is a uniform bound*

$$L(\sigma + it) \ll_L t^{d_L(1-\sigma)/2+\epsilon},$$

where d_L is the degree of L.

Proof. See Theorem 6.8 in [14]. □

Lemma 3.6 (Mean value theorem for Dirichlet polynomials) *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real or complex numbers. Let $s = \sigma + it$ be a complex variable and let*

$$X(s) = \sum_{n=1}^{N} x_n n^{-s}$$

be a Dirichlet polynomial. Then, we have

$$\int_0^T |X(s)|^2 dt = \sum_{n \le N} |x_n|^2 n^{-2\sigma}(T + O(N)).$$

Proof. This is Theorem 9.1 in [4]. □

Lemma 3.7 (Wirsing) *Suppose f is a multiplicative function such that*

(1) $\sum_{p^k \le x} f(p^k)\log p = \kappa \log x + O(1)$,

(2) $\sum_{n \le x} |f(n)| \ll (\log x)^{|\kappa|}$,

where $\kappa > -\frac{1}{2}$ is a constant. Then

$$\sum_{n \le x} f(n) = c_f(\log x)^{\kappa} + O\left((\log x)^{|\kappa|-1}\right),$$

where c_f is a constant given by

$$c_f = \frac{1}{\Gamma(\kappa+1)} \prod_p \left(1 - \frac{1}{p}\right)^{\kappa} \left(1 + f(p) + f(p^2) + \cdots\right).$$

4. Asymptotic formulae

Theorem 4.1 *Let $L \in \mathcal{S}$. Suppose that the Riemann Hypothesis holds for $L(s)$ and almost all zeros of $L(s)$ are simple. Let $M = T^\theta, \theta < 1$. Then, we have*

$$S_1 = \frac{T_2 - T_1}{2\pi} \sum_{nu \le M} \frac{a^{-1}(n) x_u y_{nu}}{nu} + \mathcal{E}_1,$$

where

$$\begin{aligned}\mathcal{E}_1 =& O\left(\left\| \frac{x_n}{n^2} \right\|_1 \|y_n\|_\infty M^{2+\epsilon} + M^\epsilon \left\| \frac{x_n}{n^2} \right\|_1 \|y_n\|_1 \right) \\ &+ O\left(T^\epsilon M \left(\left\| \frac{x_n}{n} \right\|_1 \left\| \frac{y_n}{n} \right\|_1 + \|y_n\|_1 \left\| \frac{x_n}{n} \right\|_1 + \|x_n\|_1 \left\| \frac{y_n}{n} \right\|_1 \right) \right) \\ &+ O\left(T^{-\frac{d_L}{2}} \left\| \frac{y_n}{n^2} \right\|_1 (T + \sqrt{TM}) M \left(\sum_{n \le M} |x_n|^2 \right)^{1/2} \right).\end{aligned}$$

Proof. Consider the integral

$$I_R := \frac{1}{2\pi i} \int_{c+iT_1}^{c+iT_2} L(s)^{-1} X(s) Y(1-s) ds,$$

where $c = 2$. If we move the contour left to the line $\Re(s) = 1 - c$, then the residue theorem yields $I_R = S_1 - I_L + I_H$, where

$$\begin{aligned} I_L &= \frac{1}{2\pi i} \int_{1-c+iT_1}^{1-c+iT_2} L(s)^{-1} X(s) Y(1-s) ds, \\ I_H &= \frac{1}{2\pi i} \int_{1-c+iT_1}^{c+iT_1} L(s)^{-1} X(s) Y(1-s) ds - \frac{1}{2\pi i} \int_{1-c+iT_2}^{c+iT_2} L(s)^{-1} X(s) Y(1-s) ds, \end{aligned}$$

as almost all zeros of $L(s)$ are simple by assumption. From (14), we have

$$I_R = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{a^{-1}(n)}{n^c} \sum_{u \le M} \frac{x_u}{u^c} \sum_{k \le M} \frac{y_k}{k^{1-c}} \int_{T_1}^{T_2} \left(\frac{k}{nu} \right)^{it} dt := \mathcal{M}_d + \mathcal{M}_{nd},$$

where $\mathcal{M}_d$ corresponds to the diagonal terms $k = nu$ and where $\mathcal{M}_{nd}$ corresponds to the off-diagonal terms $k \neq nu$. For the diagonal terms, $k = nu$, we have a contribution of

$$\mathcal{M}_d = \frac{T_2 - T_1}{2\pi} \sum_{nu \le M} \frac{a^{-1}(n) x_u y_{nu}}{nu}.$$

For $x \neq 1$ we have $\int_{T_1}^{T_2} x^{it} dt = O(\log |x|)^{-1}$. Thus for the off-diagonal terms, $k \neq nu$, we have

$$|\mathcal{M}_{nd}| \le \sum_{n \ge 1} \frac{|a^{-1}(n)|}{n^c} \sum_{u \le M} \frac{|x_u|}{u^c} \sum_{k \le M} \frac{|y_k|}{k^{1-c}} \frac{1}{|\log(k/nu)|}.$$

Since $c=2$, the terms for which $nu > 2M$ are bounded by

$$\sum_{n\geq 1}\frac{|a^{-1}(n)|}{n^c}\sum_{u\leq M}\frac{|x_u|}{u^c}\sum_{k\leq M}\frac{|y_k|}{k^{1-c}} \ll \sum_{n\geq 1}\frac{|a^{-1}(n)|}{n^2}\left\|\frac{x_u}{u^2}\right\|_1 \|y_n\|_1 M. \tag{15}$$

The remaining terms are bounded by

$$\sum_{nu\leq 2M}\frac{|a^{-1}(n)||x_u|}{(nu)^c}\sum_{k\neq nu}\frac{|y_k|}{k^{1-c}}\frac{1}{|\log(k/nu)|}$$

$$\ll \sum_{n\leq M}\frac{|a^{-1}(n)|}{n^2}\left\|\frac{x_u}{u^2}\right\|_1 M\|y_n\|_\infty \sup_{j\leq 2M}\left(\sum_{\substack{k\leq M\\ k\neq j}}\frac{1}{|\log(k/j)|}\right). \tag{16}$$

It suffices to bound the sum

$$\sum_{\substack{k\leq M\\ k\neq j}}\frac{1}{|\log(k/j)|}, \quad j\leq 2M.$$

The contribution from terms such that $k\leq j/2$ or $k\geq 2j$ is $O(M)$. The terms $1/2\leq k/j\leq 2$ contribute at most

$$\sum_{\max(1,j/2)\leq k\leq j-1}\frac{j}{j-k}+\sum_{j+1\leq k\leq \min(M,2j)}\frac{k}{k-j} \ll M\log M. \tag{17}$$

Combining (15), (16), and (17) with Lemma 3.2, we have

$$I_R=\frac{T_2-T_1}{2\pi}\sum_{nu\leq M}\frac{a^{-1}(n)x_u x_{nu}}{nu}+O\left(\left\|\frac{x_n}{n^2}\right\|_1\|y_n\|_\infty M^{2+\epsilon}+M^\epsilon\left\|\frac{x_n}{n^2}\right\|_1\|y_n\|_1\right).$$

Next we consider the contribution from horizontal terms. Note that

$$|X(s)Y(1-s)|=\left|\sum_{u\leq M}\frac{x_u}{u^s}\sum_{k\leq M}\frac{y_k}{k^{1-s}}\right|\leq M\left\|\frac{x_n}{n}\right\|_1\left\|\frac{y_n}{n}\right\|_1+M\|x_n\|_1\left\|\frac{y_n}{n}\right\|_1+M\|y_n\|_1\left\|\frac{x_n}{n}\right\|_1,$$

where each part corresponds to a bound for $0\leq\Re(s)\leq 1$, $-1\leq\Re(s)\leq 0$, and $1\leq\Re(s)\leq 2$ respectively. From our choice of T_1 and T_2, we have $L(\sigma+iT_j)^{-1}\ll T_j^\epsilon$. Combing these we have

$$I_H\ll T^\epsilon M\left(\left\|\frac{x_n}{n}\right\|_1\left\|\frac{y_n}{n}\right\|_1+\|x_n\|_1\left\|\frac{y_n}{n}\right\|_1+\|y_n\|_1\left\|\frac{x_n}{n}\right\|_1\right).$$

Now we estimate I_L. From (2), we write

$$L(s)=\Delta(s)_L\overline{L}(s),$$

where

$$\Delta_L(s)=\omega_L Q^{1-2s}\prod_{j=1}^{f}\frac{\Gamma\left(\lambda_j(1-s)+\overline{\mu_j}\right)}{\Gamma\left(\lambda_j s+\mu_j\right)} \tag{18}$$

Using Stirling's formula, we have for $t > 0$

$$\Delta_L(s) = \left(\lambda Q^2 t^{d_L}\right)^{\frac{1}{2}-\sigma-it} \exp\left(itd_L + \frac{i\pi(\mu - d_L)}{4}\right)\left(\epsilon + O\left(\frac{1}{|s|}\right)\right). \tag{19}$$

where $\mu = 2\sum_{j=1}^{m}(1-2\mu_j)$ and $\lambda = \prod_{j=1}^{f} \lambda_j^{2\lambda_j}$. When $\Re(s) = 1-c$, we have

$$|\Delta_L(s)| = O\left(T^{-\frac{d_L}{2}}\left(1 + O\left(\frac{1}{T}\right)\right)\right). \tag{20}$$

From Lemma 3.2, when $\Re(s) = 1-c$, we have

$$|L(1-s)| \ll 1. \tag{21}$$

From (20) and (21), we have

$$\begin{aligned} I_L &\ll T^{-\frac{d_L}{2}} \left\|\frac{y_n}{n^2}\right\|_1 \int_{T_1}^{T_2} |X(1-c+it)|dt \\ &\ll T^{-\frac{d_L}{2}} \left\|\frac{y_n}{n^2}\right\|_1 T^{1/2} \left(\int_{T_1}^{T_2} |X(1-c+it)|^2 dt\right)^{1/2} \\ &\ll T^{-\frac{d_L}{2}} \left\|\frac{y_n}{n^2}\right\|_1 T^{1/2} \left(T\sum_{n\le M} |nx_n|^2 + M\sum_{n\le M}|nx_n|^2\right)^{1/2} \\ &\ll T^{-\frac{d_L}{2}} \left\|\frac{y_n}{n^2}\right\|_1 (T+\sqrt{TM})M\left(\sum_{n\le M}|x_n|^2\right)^{1/2}, \end{aligned}$$

where we applied the Cauchy-Schwarz inequality and Lemma 3.6. This completes the estimation for S_1. □

Theorem 4.2 *Let $L \in \mathcal{S}$. Suppose $X(s) = \sum_{n\le M} \frac{x_n}{n^s}, Y(s) = \sum_{n\le M}\frac{y_n}{n^s}$, $M \le T$. Then, we have*

$$\begin{aligned} S_0 = &\left(\frac{1}{2\pi}\int_{T_1}^{T_2} \log(\lambda Q^2 t^{d_L})dt\right)\sum_{m\le M}\frac{x_m y_m}{m} \\ &- \frac{T_2 - T_1}{2\pi}\sum_{m\le M}\frac{(\Lambda_L * x)(m)y_m + \overline{\Lambda_L} * y(m)x_m}{m} + \mathcal{E}_0, \end{aligned}$$

where

$$\begin{aligned} \mathcal{E} = &O\left((\log T)^2\left(M\left\|\frac{x_n}{n}\right\|_1\left\|\frac{y_n}{n}\right\|_1 + M^{\theta_L+\epsilon}\|x_n\|_1\left\|\frac{y_n}{n}\right\|_1 + M^{\theta_L+\epsilon}\|y_n\|_1\left\|\frac{x_n}{n}\right\|_1\right)\right) \\ &+ O\left((\log T)^2 M^{1+\theta_L+\epsilon}\left(\left\|\frac{x_n}{n}\right\|_1\|y_n\|_\infty + \left\|\frac{y_n}{n}\right\|_1\|x_n\|_\infty\right)\right). \end{aligned}$$

Proof. From the residue theorem, we have

$$\begin{aligned} S_0 &= \frac{1}{2\pi i}\left(\int_{c+iT_1}^{c+iT_2} + \int_{c+iT_2}^{1-c+iT_2} + \int_{1-c+iT_1}^{c+iT_1} + \int_{1-c+iT_2}^{1-c+iT_1}\right) X(s)Y(1-s)\frac{L'}{L}(s)ds \\ &= J_R - J_L + J_H, \end{aligned}$$

where

$$
\begin{aligned}
J_R &= \frac{1}{2\pi i}\int_{c+iT_1}^{c+iT_2} X(s)Y(1-s)\frac{L'}{L}(s)ds,\\
J_L &= \frac{1}{2\pi i}\int_{1-c+iT_1}^{1-c+iT_2} X(s)Y(1-s)\frac{L'}{L}(s)ds,\\
J_H &= \frac{1}{2\pi i}\left(\int_{c+iT_2}^{1-c+iT_2} + \int_{1-c+iT_1}^{c+iT_1}\right) X(s)Y(1-s)\frac{L'}{L}(s)ds.
\end{aligned}
$$

Let $T_1 = T + O(1)$ and $T_2 = 2T + O(1)$ be such that

$$
\frac{L'}{L}(\sigma + iT_1) \ll (\log T_1)^2,\ \frac{L'}{L}(\sigma + iT_2) \ll (\log T_2)^2,
$$

uniformly for $\sigma \in [-1, 2]$. Note that

$$
\begin{aligned}
|X(s)Y(1-s)| &= \left|\sum_{u\le M}\frac{x_u}{u^s}\sum_{k\le M}\frac{y_k}{k^{1-s}}\right|\\
&\le M\left\|\frac{x_n}{n}\right\|_1\left\|\frac{y_n}{n}\right\|_1 + M^{c-1}\|x_n\|_1\left\|\frac{y_n}{n}\right\|_1 + M^{c-1}\|y_n\|_1\left\|\frac{x_n}{n}\right\|_1, \qquad (22)
\end{aligned}
$$

where each part corresponds to a bound for $0 \le \Re(s) \le 1$, $1 - c \le \Re(s) \le 0$, and $1 \le \Re(s) \le c$ respectively. Thus,

$$
J_H \ll (\log T)^2\left(M\left\|\frac{x_n}{n}\right\|_1\left\|\frac{y_n}{n}\right\|_1 + M^{c-1}\|x_n\|_1\left\|\frac{y_n}{n}\right\|_1 + M^{c-1}\|y_n\|_1\left\|\frac{x_n}{n}\right\|_1\right). \qquad (23)
$$

Taking logarithmic derivative of the functional equation (2), we have

$$
\frac{L'}{L}(s) = \frac{\Delta_L'}{\Delta_L}(s) - \frac{\overline{L'}}{\overline{L}}(1-s).
$$

Therefore,

$$
\begin{aligned}
J_L &= \frac{1}{2\pi i}\int_{1-c+T_1}^{1-c+iT_2} X(s)Y(1-s)\frac{L'}{L}(s)ds \\
&= \frac{1}{2\pi}\int_{T_1}^{T_2} X(1-c+it)Y(1-c-it)\frac{L'}{L}(1-c+it)ds \\
&= -\frac{1}{2\pi}\int_{-T_1}^{-T_2} X(1-c-it)Y(1-c+it)\frac{L'}{L}(1-c-it)dt \\
&= \overline{-\frac{1}{2\pi}\int_{-T_1}^{-T_2} \overline{X}(1-c+it)\overline{Y}(1-c-it)\frac{\overline{L}'}{\overline{L}}(1-c+it)dt} \\
&= \overline{\frac{1}{2\pi}\int_{T_1}^{T_2} \overline{X}(1-c-it)\overline{Y}(1-c+it)\frac{\overline{L}'}{\overline{L}}(1-c-it)dt} \\
&= \overline{\frac{1}{2\pi i}\int_{c+iT_1}^{c+iT_2} \overline{X}(1-s)\overline{Y}(s)\frac{\overline{L}'}{\overline{L}}(1-s)ds} \\
&= \overline{\frac{1}{2\pi i}\int_{c+iT_1}^{c+iT_2} \left\{\frac{\Delta_L'}{\Delta_L}(1-s)-\frac{L'}{L}(s)\right\}\overline{X}(1-s)\overline{Y}(s)ds}
\end{aligned}
$$

If $X(s)=\overline{Y}(s)$, then we have

$$
J_L = K-\overline{I_R},
$$

where $K=\overline{\frac{1}{2\pi i}\int_{c+iT_1}^{c+iT_2}\frac{\Delta_L'}{\Delta_L}(1-s)\overline{Y}(s)\overline{X}(1-s)ds}$. From Striling's formula, we have

$$
\frac{\Delta_L'}{\Delta_L}(s) = -\log\left(\lambda Q^2\log|t|^{d_L}\right)+O\left(\frac{1}{|t|}\right),
$$

and thus by (22),

$$
\begin{aligned}
K =& -\frac{1}{2\pi}\int_{T_1}^{T_2}\log\left(\lambda Q^2|t|^{d_L}\right)|X(c+it)|^2dt \\
&+O\left(\log T\left(M\left\|\frac{x_n}{n}\right\|_1\left\|\frac{y_n}{n}\right\|_1+M^{c-1}\|x_n\|_1\left\|\frac{y_n}{n}\right\|_1+M^{c-1}\|y_n\|_1\left\|\frac{x_n}{n}\right\|_1\right)\right). \quad (24)
\end{aligned}
$$

The main term in K denoted by K_0 is given by

$$
\begin{aligned}
K_0 &= -\frac{1}{2\pi}\int_T^{2T}\log(\lambda Q^2t^{d_L})\sum_{u\le M}\frac{x_u}{u^{1-c+it}}\sum_{k\le M}\frac{y_k}{k^{c-it}}dt \\
&= -\frac{1}{2\pi}\sum_{u\le M}\frac{x_u}{u^{1-c}}\sum_{k\le M}\frac{y_k}{k^c}\int_{T_1}^{T_2}\log\left(\lambda Q^2t^{d_L}\right)\left(\frac{k}{u}\right)^{it}dt \\
&= K_d+K_{nd}, \quad (25)
\end{aligned}
$$

where K_d denotes the contribution from the diagonal terms with $k = u$, and K_{nd} denotes the contribution from the off-diagonal terms $k \neq u$. We have

$$\begin{aligned} K_d &= -\frac{1}{2\pi} \sum_{u \leq M} \frac{x_u y_u}{u} \int_{T_1}^{T_2} \log\left(\lambda Q^2 t^{d_L}\right) dt \\ &= -\left(\frac{d_L}{2\pi} T \log T + O(T)\right) \sum_{u \leq M} \frac{x_u y_u}{u}. \end{aligned} \tag{26}$$

For K_{nd}, we have

$$\begin{aligned} K_{nd} &\ll \sum_{\substack{u,k \leq M \\ u \neq k}} \frac{x_u y_k}{u^{1-c} k^c} \frac{\log T}{|\log k/u|} \\ &\ll \log T M^{c-1} \sum_{u \leq M} |x_u| \sum_{k \leq M} \frac{|y_k|}{k^c} + \log T \sum_{u \leq M} |x_u| \sum_{u/2 \leq k \leq 2u} \frac{|y_k|}{k^c} \frac{u}{|k-u|} \\ &\ll \log T M^{c-1} \|x_n\|_1 \left\| \frac{y_k}{k^c} \right\|_1 + \log T \|x_n\|_1 \|y_n\|_\infty \log M. \end{aligned} \tag{27}$$

For J_R, we have

$$\begin{aligned} J_R &= \frac{1}{2\pi i} \int_{c+iT_1}^{c+iT_2} X(s) Y(1-s) \frac{L'}{L}(s) ds \\ &= \frac{1}{2\pi} \int_{T_1}^{T_2} \sum_{u \leq M} \frac{x_u}{u^{c+it}} \sum_{k \leq M} \frac{y_k}{k^{1-c-it}} \sum_{n=1}^{\infty} \frac{\Lambda_L(n)}{n^{c+it}} dt \\ &= J_d + J_{nd}, \end{aligned} \tag{28}$$

where J_d denotes the contribution from the diagonal terms $k = nu$, and J_{nd} denotes the contribution from the off-diagonal terms $k \neq nu$.

$$J_d = \frac{T_2 - T_1}{2\pi} \sum_{n=1}^{\infty} \sum_{u \leq M} \frac{\Lambda_L(n) x_u y_{nu}}{nu}, \tag{29}$$

and similarly to (16) and (17),

$$\begin{aligned} J_{nd} &\ll \log T \sum_{n=1}^{\infty} \frac{\Lambda_L(n)}{n^c} \sum_{u \leq M} \frac{|x_u|}{u^c} \sum_{k \leq M} \frac{|y_k|}{k^{1-c}} \frac{1}{\log |k/nu|} \\ &\ll \log T M^{c-1} \sum_{n=1}^{\infty} \frac{\Lambda_L(n)}{n^c} \left\| \frac{x_n}{n^c} \right\|_1 \|y_n\|_1 + \log T \sum_{n \leq M} \frac{\Lambda_L(n)}{n^c} \left\| \frac{x_n}{n^c} \right\|_1 M^{c-1} \|y_n\|_\infty M \log M. \end{aligned} \tag{30}$$

Taking $c = 1 + \theta_L + \epsilon$, and combining (23), (24), (26), (27), (28), (29), and (30), we complete the proof. □

5. Proof of Theorem 1.1

Let $x_n = \mu(n)a(n)$ and $y_n = \overline{x_n}$. Since x_n is supported on squarefree integers and $|a(p)| = |b(p)| \ll p^{\theta_L}$, it follows that $|x_n| \ll n^{\theta_L}$ and $\|x_n\|_1 \le M^{1+\theta_L}$. From the assumption of (5), we have

$$\sum_{p\le M} \frac{|a(p)|^2 \log p}{p} = \kappa \log M + O(1),$$

and

$$\sum_{n\le M} \frac{|\mu(n)a(n)|^2}{n} = \prod_{p\le M}\left(1+\frac{|a(p)|^2}{p}\right) \ll \exp\left(\sum_{p\le M}\frac{|a(p)|^2}{p}\right) \ll (\log M)^{\kappa}.$$

Thus from Lemma 3.7, we have

$$\sum_{n\le M} \frac{|\mu(n)a(n)|^2}{n} = (c_L + o(1))(\log M)^{\kappa}, \tag{31}$$

where

$$c_L = \frac{1}{\Gamma(\kappa+1)} \prod_p \left(1-\frac{1}{p}\right)^{\kappa} (1+|a(p)|^2). \tag{32}$$

We also have

$$\begin{aligned}
&\sum_{m\le M} \frac{(\Lambda_L * x)(m) y_m}{m} \\
&= -\sum_{m\le M} \frac{|\mu(m)a(m)|^2}{m} \sum_{\substack{p\nmid M\\ p\le M/m}} \frac{|a(p)|^2\log p}{p} \\
&= -\sum_{m\le M} \frac{|\mu(m)a(m)|^2}{m} \left(\sum_{p\le M/m} \frac{|a(p)|^2\log p}{p} - \sum_{\substack{p\mid m\\ p\le M/m}} \frac{|a(p)|^2\log p}{p} \right).
\end{aligned} \tag{33}$$

The second term in (33) can be bounded by

$$\begin{aligned}
&\sum_{m\le M} \frac{|\mu(m)a(m)|^2}{m} \sum_{\substack{p\mid m\\ p\le M/m}} \frac{|a(p)|^2\log p}{p} \\
&= \sum_{p\le M} \frac{|a(p)|^2\log p}{p^2} \sum_{\substack{m\le M\\ p\nmid m}} \frac{|\mu(m)a(m)|^2}{m} \ll \log M.
\end{aligned}$$

For the main term in (33), after applying (5) and partial summation, we have

$$\begin{aligned}&\sum_{m\leq M}\frac{|\mu(m)a(m)|^2}{m}\sum_{p\leq M/m}\frac{|a(p)|^2\log p}{p}\\ =&\kappa\sum_{m\leq M}\frac{|\mu(m)a(m)|^2}{m}\log(M/m)+O(\log M)\\ =&\frac{\kappa c_L+o(1)}{\kappa+1}(\log M)^{\kappa+1}+O((\log M)^{\kappa+1}).\end{aligned}$$

An estimate for $\sum_{m\leq M}\frac{\overline{\Lambda_L}*y(m)x_m}{m}$ can be calculated in a similar way. Thus, from Theorem 4.2, we have

$$\begin{aligned}S_0=&\frac{d_L(T_2\log T_2-T_1\log T_1)}{2\pi}(c_L+o(1))(\log M)^{\kappa}+\frac{(T_2-T_1)}{\pi}\frac{\kappa c_L+o(1)}{\kappa+1}(\log M)^{\kappa+1}\\ &+O(M^{1+2\theta_L}T^{\epsilon}).\end{aligned}$$

Applying Theorem 4.1 and (31), we have

$$\begin{aligned}S_2=&\frac{T_2-T_1}{2\pi}\sum_{n\leq M}\frac{|\mu(n)a(n)|^2}{n}+O(M^{2+\theta_L+\epsilon}+T^{\epsilon}M^{2+2\theta_L}+M^{2+\theta_L}T^{1-\frac{d_L}{2}})\\ =&\frac{T_2-T_1}{2\pi}(c_L+o(1))(\log M)^{\kappa}+O\left(M^{2+\theta_L+\epsilon}+T^{\epsilon}M^{2+2\theta_L}+M^{2+\theta_L}T^{1-\frac{d_L}{2}}\right).\end{aligned}$$

Choosing $M=T^{\theta}$ with $\theta<1/(2+\theta_L)-\epsilon$, we find that

$$\begin{aligned}S_0&=(c_L+o(1))\frac{\theta^{\kappa}}{2\pi}\left(d_L+\frac{2\kappa\theta}{\kappa+1}+o(1)\right)T(\log T)^{\kappa+1},\\ S_2&=(c_L+o(1))\frac{\theta^{\kappa}}{2\pi}T(\log T)^{\kappa+1}.\end{aligned}$$

Therefore, from (12),

$$\begin{aligned}\sum_{T_1\leq\Im\rho\leq T_2}\frac{1}{|L'(\rho)|}\geq\frac{|S_2|^2}{S_0}&\geq\frac{(c_L+o(1))\theta^{\kappa}T^2(\log T)^{2\kappa}}{2\pi(d_L+\frac{2\kappa\theta}{\kappa+1}+o(1))T(\log T)^{\kappa+1}}\\ &\geq\left(\frac{c_L\theta^{\kappa}}{2\pi(d_L+\frac{2\kappa\theta}{\kappa+1})}-o(1)\right)T(\log T)^{\kappa-1},\end{aligned}$$

where $M=T^{\theta}$ and $\theta<2/5$ is a valid choice.

6. Proof of Theorem 1.2

Proof of Theorem 1.2. Let $y_n=\bar{x}_n$ and $x_p=-a(p)f(p)$, where $f(p)$ is a multiplicative function supported on squarefree integers. Define

$$f(p)=\begin{cases}\frac{L_1}{\log p}, & \text{if } p\in[L_1^2,L_2],\\ 0, & \text{otherwise},\end{cases}\tag{34}$$

where $L_1 = \sqrt{\kappa^{-1}\log M \log\log M}$ and $L_2 = \exp((\log L_1)^2)$.

$$\sum_{n\le M}\left|\frac{x_n}{n}\right| \le \prod_{\le p\le M}\left(1+\frac{|a(p)|f(p)}{p}\right) \le \exp\left(\sum_{p\le M}\frac{|a(p)f(p)|}{p}\right). \tag{35}$$

From (34), the above becomes

$$\begin{aligned} L_1 \sum_{L_1\le p\le L_2}\frac{|a(p)|}{p\log p} &= L_1\int_{L_1}^{L_2}\frac{1}{x\log x}dA(x) \\ &= L_1\frac{A(x)}{x\log x}\bigg|_{L_1}^{L_2} + L_1\int_{L_1}^{L_2}\frac{A(x)(\log x+1)}{x^2(\log x)^2}dx \\ &\ll \sqrt{\kappa}\frac{L_1}{\sqrt{\log L_1}}, \end{aligned} \tag{36}$$

where $A(x)=\sum_{p\le x}|a(p)|$, and the last inequality follows from (6) and the fact that $A(x)\ll x^{1/2}\left((\kappa+o(1))\frac{x}{\log x}\right)^{1/2} \ll \frac{x}{\sqrt{\log x}}$. From (35) and (36), we have

$$\sum_{n\le M}\left|\frac{x_n}{n}\right| \le \exp\left(c\sqrt{\kappa\log M}\right), \tag{37}$$

and thus

$$\sum_{n\le M}|x_n| \le M\sum_{n\le M}\frac{|x_n|}{n} \le M\exp\left(c\sqrt{\kappa\log M}\right) \ll M^{1+\epsilon}, \tag{38}$$

$$\sum_{n\le M}|x_n|^2 \le M^2\sum_{n\le M}\frac{|x_n|^2}{n^2} \le M^2\exp\left(2c\sqrt{\kappa\log M}\right) \ll M^{2+\epsilon}. \tag{39}$$

Applying the bounds in (37), (38), and (39) in Theorem 4.1 and Theorem 4.2, we have

$$S_1 = \frac{(T_2-T_1)}{2\pi}\sum_{nu\le M}\frac{a^{-1}(n)x_u y_{nu}}{nu} + O\left(M^{5/2+\theta_L+\epsilon}T^{\epsilon} + T^{1-\frac{d_L}{2}+\epsilon}M^{2+\epsilon}\right),$$

$$\begin{aligned} S_0 = \frac{d_L}{2\pi}(T_2\log T_2 - T_1\log T_1)\sum_{m\le M}\frac{|x_m|^2}{m} - \frac{T_2-T_1}{2\pi}\sum_{m\le M}\frac{(\Lambda_L * x)(m)y_m + (\overline{\Lambda_L}*y)(m)x_m}{m} \\ + O\left(M^{3/2+2\theta_L+\epsilon}T^{\epsilon}\right). \end{aligned}$$

For the second sum in S_0, we have

$$
\begin{aligned}
\sum_{m\le M}\frac{(\Lambda_L * x)(m)y_m}{m} &= \sum_{p\le L_2}\frac{\Lambda_L(p)y_p}{p}\sum_{m\le M/p}\frac{|x_m|^2}{m}\\
&\ll \sum_{p\le M}\frac{b(p)\log p\ \overline{a(p)}f(p)}{p}\sum_{m\le M}\frac{|x_m|^2}{m}\\
&\ll L_1\sum_{p\le M}\frac{b(p)\log p\ \overline{a(p)}}{p\log p}\sum_{m\le M}\frac{|x_m|^2}{m}\\
&\ll L_1\kappa\log_2 M\sum_{m\le M}\frac{|x_m|^2}{m}\\
&\ll (\kappa\log M)^{1/2+\epsilon}\sum_{m\le M}\frac{|x_m|^2}{m}, \qquad (40)
\end{aligned}
$$

since we have $b(p)=a(p)$ and are assuming (7). Choosing $M^{5/2+\theta_L+\epsilon}\ll T$, and using (40), we have

$$
S_0=\left(\frac{d_L}{2\pi}T\log T+o(1)\right)\sum_{m\le M}\frac{|a_m|^2f(m)^2}{m}.
$$

Since x_n is supported on squarefree integers, we have

$$
\begin{aligned}
a^{-1}(n)x_u y_{nu} &= \mu(n)a(n)\mu(u)a(u)f(u)\mu(nu)\overline{a(nu)}f(nu)\\
&= |a(n)a(u)|^2f(u)f(nu),
\end{aligned}
$$

and it follows that

$$
\frac{|S_1|}{S_0}\gg\sum_{nu\le M}\frac{|a(n)a(u)|^2f(n)f(nu)}{nu}\Big/\left(\log T\sum_{m\le M}\frac{|a(m)|^2f(m)^2}{m}\right). \qquad (41)
$$

Since $f(n)$ is multiplicative and supported on squarefree numbers,

$$
\begin{aligned}
&\sum_{nu\le M}\frac{|a(n)a(u)|^2f(n)f(nu)}{nu}\\
&=\sum_{n\le M}\frac{|a(n)|^2f(n)}{n}\sum_{\substack{u\le M/n\\(u,n)=1}}\frac{|a(u)|^2f(u)^2}{u}\\
&=\sum_{n\le M}\frac{|a(n)|^2f(n)}{n}\left(\prod_{(p,n)=1}\left(1+\frac{|a(p)|^2f(p)^2}{p}\right)-\sum_{\substack{u\ge M/n\\(n,u)=1}}\frac{|a(u)|^2f(u)^2}{u}\right).
\end{aligned}
$$

By Rankin's trick, the contribution from $u > M/n$ is bounded by

$$\sum_{n\le M}\frac{|a(n)|^2 f(n)}{n}\left(\frac{n}{M}\right)^{\alpha}\sum_{\substack{u=1\\(u,n)=1}}^{\infty}\frac{|a(u)|^2 f(u)^2 u^{\alpha}}{u}$$
$$\le \frac{1}{M^{\alpha}}\prod_p\left(1+|a(p)|^2 f(p)^2 p^{\alpha-1}+|a(p)|^2 f(p)p^{\alpha-1}\right) \tag{42}$$

for any $\alpha > 0$. By Rankin's trick again, the main term becomes

$$\prod_p\left(1+\frac{|a(p)|^2 f(p)^2}{p}+\frac{|a(p)|^2 f(p)}{p}\right)+O\left(\frac{1}{M^{\alpha}}\prod_p\left(1+\frac{|a(p)|^2 f(p)^2}{p}+\frac{|a(p)|^2 f(p)p^{\alpha}}{p}\right)\right). \tag{43}$$

Combining (43) and (42), we deduce that

$$\sum_{nu\le M}\frac{|a(n)a(u)|^2 f(u)f(nu)}{nu}=\mathcal{Q}_1+O\left(\frac{1}{M^{\alpha}}\prod_p\left(1+|a(p)|^2 f(p)^2 p^{\alpha-1}+|a(p)|^2 f(p)p^{\alpha-1}\right)\right),$$

where

$$\mathcal{Q}_1=\prod_p\left(1+\frac{|a(p)|^2 f(p)^2}{p}+\frac{|a(p)|^2 f(p)}{p}\right).$$

Note that the ratio of the error to the main term is bounded by

$$\ll \exp\left(-\alpha\log M+\sum_{L_1^2\le p\le \exp((\log L_1)^2)}|a(p)|^2(p^{\alpha}-1)\left(\frac{L_1^2}{p\log^2 p}+\frac{L_1}{p\log p}\right)\right)$$
$$\ll \exp\left(-\alpha\frac{\log M}{\log_2 M}\right).$$

Choosing $\alpha = 1/(\log L_1)^3$ yields

$$\sum_{nu\le M}\frac{|a(n)a(u)|^2 f(u)f(nu)}{nu}=\mathcal{Q}_1(1+o(1)).$$

We also have the inequality

$$\sum_{m\le M}\frac{|a(m)|^2 f(m)^2}{m}\le\sum_{n}\frac{|a(m)|^2 f(m)^2}{m}=\prod_p\left(1+\frac{|a(p)|^2 f(p)^2}{p}\right)=:\mathcal{Q}_0.$$

From the definitions of $\mathcal{Q}_0$ and $\mathcal{Q}_1$, it can be seen that

$$\frac{\mathcal{Q}_1}{\mathcal{Q}_0}=\prod_p\left(1+\frac{|a(p)|^2 f(p)}{p(1+|a(p)|^2 f(p)^2 p^{-1})}\right).$$

Since

$$\sum_{L_1 \le p \le \exp((\log L_1)^2)} \frac{|a(p)|^2 f(p)}{p(1+|a(p)|^2 f(p)^2 p^{-1})} = \sum_{L1^2 \le p \le \exp((\log L_1)^2)} \frac{L_1 |a(p)|^2}{p \log p}(1+o(1))$$
$$= (\kappa + o(1)) \frac{L_1}{\log L_1^2},$$

we have

$$\frac{\mathcal{Q}_1}{\mathcal{Q}_0} \ge \exp\left((\kappa + o(1)) \frac{L_1}{\log L_1^2} \right) = \exp\left(\sqrt{(1+o(1)) \frac{\kappa \log M}{\log \log M}} \right).$$

Therefore, from (41), we have

$$\frac{|S_1|}{S_0} \gg \exp\left((1+o(1)) \sqrt{\frac{\kappa \log M}{\log \log M}} \right).$$

□

7. Proof of Proposition 3.1

Lemma 7.1 (Theorem of Borel-Carathéodory) *Let $f(z)$ be a holomorphic function on $|z| \le R$, and let $M(r) = \sup_{|z|=r} |f(z)|$ and $A(r) = \sup_{|z|=r} \Re(f(z))$. Then, for $0 < r < R$, we have*

$$M(r) \le \frac{2r}{R-r} A(R) + \frac{R+r}{R-r} |f(0)|.$$

Lemma 7.2 (Hadamard's three circle theorem) *Let f be analytic on a region containing the set $R = \{z | r_1 \le |z| \le r_3\}$. Then, for $0 < r_1 < r_2 < r_3$, we have*

$$M_2^{\log(r_3/r_1)} \le M_1^{\log(r_3/r_2)} M_3^{\log(r_2/r_1)},$$

where $M_i = \sup_{|z|=r_i} |f(z)|$ for $i = 1, 2, 3$.

Lemma 7.3 *Suppose $f(s)$ is regular, and in the circle $|s - s_0| \le r$, we have*

$$\frac{|f(s)|}{|f(s_0)|} \le e^M, M > 1.$$

Then, for $|s - s_0| \le \frac{r}{4}$, we have

$$\left| \frac{f'(s)}{f(s)} - \sum_{|\rho - s_0| \le \frac{r}{2}} \frac{1}{s - \rho} \right| \ll \frac{M}{r},$$

where ρ runs through the zeros of $f(s)$ such that $|\rho - s_0| \le \frac{1}{2} r$.

Lemma 7.4 *Let $L \in \mathcal{S}$ and let $N_L(T)$ denote the number of zeros of $L(s)$ in the rectangle $0 \le \Re(s) \le 1$ with $0 < \Im(s) \le T$. Then,*

$$N_L(T) = \frac{d_L}{2\pi} T \log T + c_{L,1} T + c_{L,2} + \arg L(\tfrac{1}{2} + iT) + O\left(\frac{1}{T}\right),$$

where d_L is the degree of $L(s)$.

Proof.

$$2N_L(T) = \frac{2}{\pi}\Delta\Xi(s), \tag{44}$$

where Δ denotes the variation from 2 to $2+iT$ and then to $\frac{1}{2}+iT$, along straight lines. Thus

$$\pi N_L(T) = \Delta \arg Q^s + \sum_{j=1}^{f} \Delta\Gamma(\lambda_j s + \mu_j) + \Delta \arg L(s).$$

Since we have

$$\begin{aligned}\Delta Q^s &= -T\log Q,\\ \Delta\Gamma(\lambda_j s + \mu_j) &= \Im \log\Gamma(\frac{\lambda_j}{2} + i\lambda_j T + \mu_j)\\ &= \frac{\lambda_j}{2}\log(\lambda_j T) - \frac{\lambda_j}{2}T + c_j + O\left(\frac{1}{T}\right),\end{aligned}$$

the lemma follows. □

Lemma 7.5 *If $\frac{1}{2} < \alpha < \sigma < \beta$, $T < t \le T'$, then we have*

$$\log L(s) = \frac{1}{\pi}\int_{\alpha+iT}^{\alpha+iT'} \frac{\arg L(z,\pi)}{s-z}dz + O\left(\frac{\log T}{t-T}\right) + O\left(\frac{\log T'}{T'-t}\right).$$

Proof. From the residue theorem,

$$\log L(s) = \frac{1}{2\pi i}\left(\int_{\beta+iT}^{\beta+iT'} + \int_{\beta+iT'}^{\alpha+iT'} + \int_{\alpha+iT'}^{\alpha+iT} + \int_{\alpha+iT}^{\beta+iT}\right)\frac{\log L(z)}{z-s}dz.$$

Let $\beta > 2$. Since uniformly for $\frac{1}{2} < \sigma_0 \le \sigma \le 1$,

$$\log L(s) = O\left((\log t)^{2-\sigma+\epsilon}\right)$$

holds, it follows that

$$\int_{\alpha+iT}^{2+iT} \frac{\log L(z)}{z-s}dz = O\left(\frac{1}{t-T}\int_{\alpha}^{2} |\log L(x+iT)|dx\right) = O\left(\frac{\log T}{t-T}\right). \tag{45}$$

Also,

$$\begin{aligned}\int_{2+iT}^{\beta+iT} \frac{\log L(z)}{z-s}dz &= \sum_{n=2}^{\infty}\Lambda_{\pi,1}(n)\int_{2+iT}^{\beta+iT}\frac{n^{-s}}{z-s}dz\\ &= O\left(\sum_{n=1}^{\infty}\Lambda_{\pi,1}(n)\frac{1}{n^2(t-T)}\right)\\ &= O\left(\frac{1}{t-T}\right),\end{aligned} \tag{46}$$

where $\Lambda_{\pi,1}(n)$ is the coefficient of $\log L(s)$. The last equality follows from the fact that $\Lambda_{\pi,1}(n) \ll \sqrt{n}$, since

$$\log L(s) = \sum_p \sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}} \tag{47}$$

and $b(p^k) \ll p^{k\theta_L}$ for some $\theta_L < 1/2$. By (45) and (46), we have

$$\int_{\alpha+iT}^{\beta+iT} \frac{\log L(z)}{z-s} dz = O\left(\frac{\log T}{t-T}\right). \tag{48}$$

Similarly,

$$\int_{\alpha+iT'}^{\beta+iT'} \frac{\log L(z)}{z-s} dz = O\left(\frac{\log T'}{T'-t}\right), \tag{49}$$

and

$$\int_{\beta+iT}^{\beta+iT'} \frac{\log L(z)}{z-s} dz = O\left(\frac{T'-T}{\beta-\sigma}\right). \tag{50}$$

Combining (48), (49), (50) and letting $\beta \to \infty$, we have

$$\log L(s) = \frac{1}{2\pi i} \int_{\alpha+iT}^{\alpha+iT'} \frac{\log L(z)}{s-z} dz + O\left(\frac{\log T}{t-T}\right) + O\left(\frac{\log T'}{T'-t}\right). \tag{51}$$

Similarly, if $\Re(s') < \frac{1}{2}$, then

$$0 = \frac{1}{2\pi i} \int_{\alpha+iT}^{\alpha+iT'} \frac{\log L(z)}{s'-z} dz + O\left(\frac{\log T}{t-T}\right) + O\left(\frac{\log T'}{T'-t}\right). \tag{52}$$

Taking $s' = 2\alpha - \sigma + it$, so that $s' - z = \alpha - iy - (\sigma - it)$, and replacing (52) by its conjugate, we have

$$0 = \frac{1}{2\pi i} \int_{\alpha+iT}^{\alpha+iT'} \frac{\log|L(z)| - i\arg L(z)}{z-s} dz + O\left(\frac{\log T}{t-T}\right) + O\left(\frac{\log T'}{T'-t}\right). \tag{53}$$

Combining (51) and (53), we have

$$\log L(s) = \frac{1}{\pi i} \int_{\alpha+iT}^{\alpha+iT'} \frac{\log|L(z)|}{z-s} dz + O\left(\frac{\log T}{t-T}\right) + O\left(\frac{\log T'}{T'-t}\right), \tag{54}$$

$$\log L(s) = \frac{1}{\pi} \int_{\alpha+iT}^{\alpha+iT'} \frac{\arg L(z)}{z-s} dz + O\left(\frac{\log T}{t-T}\right) + O\left(\frac{\log T'}{T'-t}\right). \tag{55}$$

□

Lemma 7.6 *Let $S(t, L) = \frac{1}{\pi} \arg L(\frac{1}{2} + it)$. If $L(s)$ has no zeros when $\Re(s) > \frac{1}{2}$, then*

$$S(t, L) \ll_L \frac{\log t}{\log\log t}, \tag{56}$$

$$S_1(t, L) \ll_L \frac{\log t}{(\log\log t)^2}, \tag{57}$$

where $S_1(t, L) = \frac{1}{\pi} \int_{\frac{1}{2}}^{\infty} \log|L(\sigma + it)| d\sigma$.

Proof. This can be derived from Theorem 1 and Theorem 2 in [1]. In [1], the L-functions are restricted to those with polynomial products, but the argument only requires a bound for Λ_L of the shape $\Lambda_L(n) \le d_L \Lambda(n) n^{\theta}$. This is satisfied for $L(s) \in \mathcal{S}$, since $\Lambda_L(n) = b(n)\log n \ll \Lambda(n) n^{\theta_L+\epsilon}$. □

Lemma 7.7 *For any* $\sigma > \frac{1}{2}$, $0 < \xi < \frac{1}{2}t$,

$$\log L(s) = i\int_{t-\xi}^{t+\xi} \frac{S(y,L)}{s-\frac{1}{2}-iy}dy + O\left(\frac{\phi(2t)}{\xi}\right) + O(1), \tag{58}$$

where $\phi(t) = \max_{1\le t\le t} S_1(t,\pi)$.

Proof. From (55) with $\alpha \to \frac{1}{2}$, one has

$$\log L(s) = i\int_{\frac{1}{2}t}^{2t} \frac{S(y,L)}{s-\frac{1}{2}-iy}dy + O(1), \tag{59}$$

since $S_1(y,L) = O(\log y)$. Therefore

$$\begin{aligned}\int_{t+\xi}^{2t} \frac{S_1(y,L)}{s-\frac{1}{2}-iy}dy &= \frac{S_1(y,L)}{s-\frac{1}{2}-iy}\bigg|_{t+\xi}^{2t} - i\int_{t+\xi}^{2t}\frac{S_1(y,L)}{(s-\frac{1}{2}-iy)^2}dy \\ &= O\left(\frac{\phi(2t)}{\xi}\right) + O\left(\phi(2t)\int_{t+\xi}^{2t}\frac{dy}{(\sigma-\frac{1}{2})^2+(y-t)^2}\right) \\ &= O\left(\frac{\phi(2t)}{\xi}\right),\end{aligned}$$

and similarly for the integral over $(\frac{1}{2}t, t-\xi)$. Thus the result follows from (59). □

Lemma 7.8 *For* $\frac{1}{2} \le \sigma \le \frac{1}{2} + c\frac{\log t}{\log\log t}$, *we have*

$$-A\frac{\log t}{\log\log t}\log\left(\frac{2}{(\sigma-\frac{1}{2})\log\log t}\right) \le \log|L(s)| \le A\frac{\log t}{\log\log t}, \tag{60}$$

where A is some constant depending on L.

Proof. Taking the real part in (58), one sees that

$$\log|L(s)| = \int_0^{\xi}\frac{x}{(\sigma-\frac{1}{2})^2+x^2}\left(S(t-x,L)-S(t+x,L)\right)dx + O\left(\frac{\phi(2t)}{\xi}\right) + O(1). \tag{61}$$

From Lemma 7.4, we have

$$N_L(T) = \frac{d_L}{2\pi}T\log T + c_{L,1}T + c_{L,2} + S(T,L) + O\left(\frac{1}{T}\right).$$

Therefore,

$$S(t+x,L) - S(x,L) \ge -Ax\log t + O\left(x/t^2\right) \tag{62}$$

for some constant A depending on $L(s)$. Combining (62), (61) and (57), we obtain

$$\begin{aligned}\log|L(s)| &\le A\int_0^{\xi}\frac{x^2\log t}{(\sigma-\frac{1}{2})^2+x^2}dx+O\left(\frac{\log t}{\xi(\log\log t)^2}\right)+O(1)\\ &\le A\xi\log t+O\left(\frac{\log t}{\xi(\log\log t)^2}\right),\end{aligned}$$

uniformly for $\sigma>\frac{1}{2}$ and so by continuity, for $\sigma=\frac{1}{2}$ as well. Taking $\xi=1/\log\log t$, we have

$$\log|L(s)|\le A\frac{\log t}{\log\log t}.$$

On the other hand, from (55) and (56),

$$\log L(s)=O\left(\frac{\log t}{\log\log t}\int_0^{\xi}\frac{dx}{\sqrt{(\sigma-\frac{1}{2})^2+x^2}}\right)+O\left(\frac{\log t}{\xi(\log\log t)^2}\right)+O(1). \tag{63}$$

Also,

$$\int_0^{\xi}\frac{dx}{\sqrt{(\sigma-\frac{1}{2})^2+x^2}}=\int_0^{\xi/(\sigma-1/2)}\frac{dx}{\sqrt{1+x^2}}\le\begin{cases}1, & \text{if } \xi\le\sigma-\frac{1}{2},\\ 1+\log\frac{\xi}{\sigma-\frac{1}{2}}, & \text{otherwise.}\end{cases}.$$

Therefore, by taking $\xi=1/\log\log t$ in (63), we find that

$$\log|L(s)|\ge -A\frac{\log t}{\log\log t}\log\left(\frac{2}{(\sigma-\frac{1}{2})\log\log t}\right).$$

□

Taking $\sigma=\frac{1}{2}+\frac{c}{\log\log t}$, we obtain the following corollary.

Corollary 7.9 *Let $s=\sigma+it$. We have*

$$\log|L(s)|=O\left(\frac{\log t}{\log\log t}\right),\ \sigma=\frac{1}{2}+\frac{c}{\log\log t}. \tag{64}$$

Proof of Proposition 3.1. Let $\delta=1/\log\log T$. Then, the bound holds for $\sigma\ge\frac{1}{2}+\delta$ from (64). We therefore assume that $\frac{1}{2}\le\sigma\le\frac{1}{2}+\delta$. We apply Lemma 7.3 with $f(s)=L(s), s_0=\frac{1}{2}+\frac{1}{\sqrt{3}}\delta+iT$, and $r=\frac{4}{\sqrt{3}}\delta$. From (64), we have

$$\left|\frac{1}{L(s_0)}\right|\le\exp\left(\frac{A\log T}{\log\log T}\right).$$

From (60), we have for $|s-s_0|\le r$ and $\sigma\ge\frac{1}{2}$,

$$|L(s)|\le\exp\left(\frac{A\log T}{\log\log T}\right).$$

For $|s-s_0|\le r$ and $\sigma<\frac{1}{2}$, the functional equation gives

$$|L(s)|\ll t^{d_L(\frac{1}{2}-\sigma)}|L(1-s)|\ll\exp\left(\frac{A'\log T}{\log\log T}\right).$$

Since $s_0-\rho=\frac{1}{\sqrt{3}}\delta+i(T-\gamma)$, we have $|s_0-\rho|\le\frac{r}{2}$ if and only if $|T-\gamma|\le\delta$. It then follows from Lemma 7.3 that for $|s-s_0|\le\frac{r}{4}$, and so in particular $\frac12\le\sigma\le\frac12+\delta$, $t=T$, we have

$$\frac{L'(s)}{L(s)}=\sum_{|t-\gamma|\le\delta}\frac{1}{s-\rho}+O\left(\log T\right). \tag{65}$$

Integrating (65), we obtain

$$\log\frac{L(s)}{L(s_0)}=\sum_{|t-\gamma|\le\delta}\log\left(\frac{s-\rho}{s_0-\rho}\right)+O\left(\frac{\log T}{\log\log T}\right). \tag{66}$$

Taking the real part in (66), and combining with (64), we deduce that

$$\begin{aligned}\log|L(s)|&=\sum_{|t-\gamma|\le\delta}\log\left|\frac{s-\rho}{s_0-\rho}\right|+O\left(\frac{\log T}{\log\log T}\right)\\&\ge\sum_{|t-\gamma|\le\delta}\log\frac{|t-\gamma|}{2\delta}+O\left(\frac{\log T}{\log\log T}\right).\end{aligned}$$

Now observe that

$$\begin{aligned}\int_T^{T+1}\sum_{|t-\gamma|\le\delta}\log\frac{|t-\gamma|}{2\delta}dt&=\sum_{T-\delta\le\gamma\le T+1+\delta}\int_{\max(\gamma-\delta,T)}^{\min(\gamma+\delta,T+1)}\log\frac{|t-\gamma|}{2\delta}dt\\&\ge\sum_{T-\delta\le\gamma\le T+1+\delta}\int_{\gamma-\delta}^{\gamma+\delta}\log\frac{|t-\gamma|}{2\delta}dt\\&=\sum_{T-\delta\le\gamma\le T+1+\delta}(-2\delta-2\delta\log 2)\\&\ge -A''\delta\log T,\end{aligned}$$

as there are $O(\log T)$ such terms in the sum. Hence there is a $t\in[T,T+1]$ for which

$$\sum_{|t-\gamma|\le\delta}\log\frac{|t-\gamma|}{2\delta}\ge -A''\delta\log T,$$

which gives

$$\log|L(\sigma+it)|\ge -A'''\frac{\log t}{\log\log t}.$$

□

8. Acknowledgement

The authors are grateful to Micah Milinovich for useful comments and suggestions. The authors also would like to thank the referee for comments in improving the presentation of the paper.

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Mathematisches Institut, Georg-August Universität Göttingen, Bunsenstrasse 3-5, D-37073, Göttingen Germany

E-mail address: junxian.li@mathematik.uni-goettingen.de

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, United States and Simion Stoilow Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, RO-014700 Bucharest, Romania

E-mail address: zaharesc@illinois.edu