LARGE VALUES OF DIRICHLET L-FUNCTIONS AT ZEROS OF A CLASS OF L-FUNCTIONS

JUNXIAN LI

Abstract. In this paper, we are interested in obtaining large values of Dirichlet L-functions evaluated at zeros of a class of L-functions, that is, $\max_{\substack{F(\rho)=0\\ T\leq \Im\rho\leq 2T}} L(\rho,\chi)$, where χ is a primitive Dirichlet character and F belongs to a class of L-functions. The class we consider includes L-functions associated to automorphic representations of $GL(n)$ over $\mathbb{Q}$.

1. Introduction

The study of the value distribution of the Riemann zeta function dates back to the work of H. Bohr. Using the theory of almost periodic functions, he showed that $\zeta(s)$ takes any nonzero complex value z infinitely often in any strip $1<\Re(s)<1+\epsilon$. Later in [6], together with B. Jessen, he showed that $\log\zeta(\sigma+it)$ has a continuous limiting distribution on the complex plane for any $\sigma>\frac{1}{2}$. On the critical line, A. Selberg [42, 41] showed that $\log|\zeta(\frac{1}{2}+it)|$ is approximately Gaussian distributed in the sense that

$$\frac{1}{T}\operatorname{meas}\left\{t\in[T,2T]:\frac{\log|\zeta(\frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\log\log T}}\geq\lambda\right\}\to\frac{1}{\sqrt{2\pi}}\int_{\lambda}^{\infty}e^{-x^2/2}dx,\ \text{as } T\to\infty. \tag{1.1}$$

This implies that the typical size of $|\zeta(\frac{1}{2}+it)|$ is $\exp\left(\sqrt{\frac{1}{2}\log\log T}\right)$. Regarding the exceptional large values of $|\zeta(\frac{1}{2}+it)|$, the Lindelöf Hypothesis asserts that $|\zeta(\frac{1}{2}+it)|=o(t^{\epsilon})$ for any $\epsilon>0$. Assuming the Riemann Hypothesis, one can show ([34, 14]) that

$$|\zeta(\tfrac{1}{2}+it)|=O\left(\exp\left(c\frac{\log t}{\log\log t}\right)\right),\ \text{as } t\to\infty,$$

for some absolute constant c. D. Farmer, S. Gonek, and C. Hughes [20] conjectured that the maximum value of $\zeta(\frac{1}{2}+it)$ for t in the interval $[0,T]$ is of order $\exp\left(\left(\frac{1}{\sqrt{2}}+o(1)\right)\sqrt{\log T\log\log T}\right)$. For omega results, E. C. Titchmarsh [46, Theorem 8.12] first showed that there exist arbitrarily large t with $|\zeta(\frac{1}{2}+it)|\geq\exp(\log^{\alpha}t)$ for any $\alpha<1/2$. Under the Riemann Hypothesis, H. Montgomery [37], proved that there exist arbitrarily large values of t such that

$$|\zeta(\tfrac{1}{2}+it)|\gg\exp\left(\tfrac{1}{20}\sqrt{\frac{\log t}{\log\log t}}\right).$$

R. Balasubramanian and K. Ramachandra [2] showed unconditionally that there are arbitrarily large t such that

$$|\zeta(\tfrac{1}{2}+it)|\gg\exp\left(c\sqrt{\frac{\log t}{\log\log t}}\right), \tag{1.2}$$

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for some positive constant c. Later, K. Soundararajan [43] introduced the resonance method and obtained (1.2) for $c = 1 + o(1)$. More recently, A. Bondarenko and K. Seip in a series of papers [9, 10, 11] proved that for any $0 \le \beta < 1$ and $0 \le c \le \sqrt{1-\beta}$, if T is sufficiently large, then

$$\max_{T^\beta \le t \le T} |\zeta(\tfrac{1}{2} + it)| \gg \exp\left(c\sqrt{\frac{\log T \log\log\log T}{\log\log T}} \right). \tag{1.3}$$

The constant c has been further improved by R. de la Bretèche and G. Tenenbaum [19] by a factor of $\sqrt{2}$. Fewer results have been investigated on large values of degree $\zeta(s)$ at discrete points on the critical line. X. Li and M. Radziwiłł[33] considered the large values of $\zeta(\frac{1}{2} + it)$ in vertical arithmetic progressions on the critical line. J. Kalpokas and P. Sarka [31] considered large values at generalized Gram points. In this paper, we consider the large values of the Riemann zeta function and Dirichlet L-functions at the zeros of a class of L-functions.

Theorem 1.1. *Let χ be a primitive Dirichlet L-functions with conductor $q > 1$. If all non-trivial zeros of $L(s, \chi)$ are on the critical line $\Re(s) = \frac{1}{2}$, then for T sufficiently large,*

$$\max_{\substack{L(\rho,\chi)=0 \\ T \le \Im\rho \le 2T}} |\zeta(\rho)| \gg \exp\left(c\sqrt[4]{\frac{\log T}{\phi(q)(\log\log T)^2}} \right),$$

where c is some absolute positive constant.

This can be improved if we assume the Riemann Hypothesis holds for all Dirichlet L-functions.

Theorem 1.2. *Let χ and ψ be two different primitive Dirichlet characters. Under the assumption that the Riemann Hypothesis is true for all Dirichlet L-functions,*

$$\max_{\substack{L(\rho,\psi)=0 \\ T \le \Im\rho \le 2T}} |L(\rho, \chi)| \gg \exp\left(c\sqrt{\frac{\log T}{\phi(d)\log\log T}} \right)$$

for some $c > 0$, where d is the least common multiple of the conductors of χ and ψ. [1]

It is believed that the values of distinct primitive L-functions are uncorrelated. For example, it is conjectured that different primitive Dirichlet L-functions have no common non-trivial zeros ([23, Conjecture 3]). A. Fujii [22] proved this is true for a positive proportion of distinct primitive Dirichlet characters. Under the Riemann Hypothesis, B. Conrey, A. Ghosh, and S. Gonek [15, 16] showed that at most two-thirds of the zeros of $\zeta(s)$ are also zeros of $L(s, \chi)$, where χ is a non-principal Dirichlet character. They remarked in [16] that similar results hold for Dirichlet L-functions with inequivalent characters under the Generalized Riemann Hypothesis. R. Garunkštis and J. Kalpokas [24] gave a lower bound for the proportion uniformly in the size of the conductors of the characters. Our result shows that under GRH, the values of $\zeta(s)$ at the zeros of another primitive L-function can be almost as large as the extreme large values of $\zeta(s)$ on the critical line without constraints.

Even though we were not able to obtain a bound as good as in (1.3) for individual L-functions, we can show a bound of the same shape as in (1.3) for the value $|\zeta_K(s)|$, the Dedekind zeta function associated to a number field K, on the critical line. When $K = \mathbb{Q}(\zeta_n)$, we know that $\zeta_K(s) = \prod_{\chi \pmod n} L(s, \chi)$. For a general number field, $\zeta_K(s)$ can be factored into Artin L-functions associated to irreducible representations of $\mathrm{Gal}(K/\mathbb{Q})$. The Langlands reciprocity conjecture implies that each factor is an L-function for an irreducible cuspidal automorphic representation π of $GL(m)$ over $\mathbb{Q}$. Thus it make sense to study the values of $\zeta(s)$ at the zeros of automoprhic L-functions. We give a result in this direction.

[1] Under a weaker larger zero free region assumption, it was claimed [38, Theorem 1.2] that there exists of large value of $\zeta'(\rho)$ of size $\exp\left(c\sqrt{\frac{\log|\Im\rho|}{\log\log|\Im\rho|}}\right)$. However, the argument is problematic as it based on an upper bound for $\mathcal{H}(z)$, whose derivation missed a factor of $\max_{q \le z} |y_q|$, which too big for the application.

Theorem 1.3. *Let $m \geq 2$ and π be an irreducible automorphic representation of $GL(m)$ over $\mathbb{Q}$. Assuming that $L(s,\pi)$ has all its non-trivial zeros on the line $\Re(s) = \frac{1}{2}$, then for sufficiently large T*

$$\max_{\substack{L(\rho,\pi)=0\\ T\leq T\leq 2T}} |\zeta(\rho)| \gg \exp\left(c_1 \sqrt[4]{\frac{\log T}{(\log\log T)^2}}\right),$$

for some positive constant c_1 depending on the conductor of π. Let $\chi \pmod q$ be a Dirichlet character such that $L(s,\pi\otimes\chi)$ has no pole at $s=1$. Then under the Grand Riemann Hypothesis, we have for sufficiently large T

$$\max_{\substack{L(\rho,\pi)=0\\ T\leq \Im\rho\leq 2T}} |L(\rho,\chi)| \gg \exp\left(c_2 \sqrt{\frac{\log T}{\log\log T}}\right),$$

where $c_2 > 0$ is some positive constant depending on π and χ.

As a corollary, we have

Theorem 1.4. *Let f be a holomorphic primitive cusp form of weight $k \geq 1$, level q and let χ be a primitive Dirichlet character. If $L(f,s)$ has all non-trivial zeros on the critical line $\Re(s) = \frac{1}{2}$, then for T large enough,*

$$\max_{\substack{L(f,\rho)=0\\ T\leq \Im\rho\leq 2T}} |\zeta(\rho)| \gg \exp\left(c_3 \sqrt[4]{\frac{\log T}{(\log\log T)^2}}\right),$$

where $c_3 > 0$ is some positive constant depending on f and χ.

Automorphic L-functions are conjectured to belong to the Selberg class. The Riemann zeta function and Dirichlet L-functions are examples of degree 1 L-functions from the Selberg class. Many results mentioned above have been generalized to L-functions in the Selberg class with additional conditions. For example, E. Bombieri and D. Hejhal [7] proved that $\{\log(L_j(\frac{1}{2}+it))\}_{j=1}^N$ behave like independent Gaussian distributed random variables for certain L_j's in the Selberg class. A short interval analogue was proved by E. Bombieri and A. Perelli [8]. In the same paper [8], they also considered the simultaneous non-vanishing in the setting of the Selberg class under certain additional hypotheses. Some unconditional results for cuspidal automorphic representations have been established by R. Raghunathan [40]. In terms of large values of L-functions in the Selberg class, C. Aistleitner and L. Pańkowaski [1] have some results for L-functions in the Selberg class with polynomial Euler products. Our result could apply to L-functions in the Selberg class with some additional conditions (see Section 3).

2. Outline

We prove a general theorem for a class of functions $\mathcal{S}^*$. Theorem 1.1- Theorem 1.4 will then follow. The idea is to use the resonance method to compute

$$S_1 = \sum_{\substack{F(\rho)=0\\ T_1\leq \Im\rho\leq T_2}} L(\rho,\chi)X(\rho)Y(1-\rho), \tag{2.1}$$

$$S_0 = \sum_{\substack{F(\rho)=0\\ T_1\leq \Im\rho\leq T_2}} X(\rho)Y(1-\rho), \tag{2.2}$$

where $F(s)$ is an L-function in $\mathcal{S}$, $L(s,\chi)$ is a Dirichlet L-function, $T_1 = T+O(1)$ and $T_2 = 2T+O(1)$ are chosen to be $\gg 1/\log T$ away from the zeros of F, $X(s) = \sum_{n\leq M} \frac{x_n}{n^s}$ and $Y(s) = \sum_{n\leq M} \frac{y_n}{n^s}$. If $\Re\rho = \frac{1}{2}$ and $y_n = \overline{x_n}$, then $X(\rho)Y(1-\rho) = |X(\rho)|^2$ and thus

$$\max_{\substack{F(\rho)=0\\ T\leq \Im\rho\leq 2T}} |L(\rho,\chi)| \geq \frac{|S_1|}{S_0}.$$

To compute the values of $L(\rho,\chi)$ at zeros of F, we use the method of Conrey, Ghosh and Gonek [17] in the study of simple zeros of $\zeta(s)$. To this end, we need some additional conditions on F and thus we restrict ourselves to a subclass $\mathcal{S}^*$. This was also used by Ng [39] in studying extreme values of $\zeta'(\rho)$. The size of the resonator requires a large zero free region so that the error terms are negligible. When taking F to be a Dirichlet L-function, the classical zero free region allows one to take $M = \exp\left(c\sqrt{\log T}\right)$ for some positive constant c if there are no Siegel zeros. Even though non-existence of Siegel zeros is still an open problem, we do know that Siegel zeros are very rare if they exist. If we assume the non-trivial zeros of all L-functions $\mathcal{S}$ are on the line $\Re(s) = \frac{1}{2}$, we can take the length of the resonator M to be T^c for some positive constant c under the Ramanujan Conjecture. This will give a bound of the form as in (1.2). An essential part is related to the study of the coefficients of $\frac{F'}{F}(s)$ in arithmetic progressions. We employ a variant of Perron's formula by J. Liu and Y. Ye [35] to avoid assuming the Generalized Ramanujan Conjecture on the coefficients of $F(s)$.

The organization of the rest of the paper is as follows. In Section 3, we define a class of L-functions $\mathcal{S}$ and its subclass $\mathcal{S}^*$ and give their properties. In Section 4, we show that L-functions associated to irreducible cuspidal representations of $GL(n)$ belong to $\mathcal{S}^*$. In Section 5, we give an estimate of S_0 as defined in (2.2). In Section 6, we give the asymptotic for S_1 as defined in (2.1) for $F \in \mathcal{S}^*$. In Section 7, we define the resonator coefficients and give some properties of the resonator. In Section 8, we complete the proof of Theorem 1.1-Theorem 1.4. Throughout the paper c, β, ϵ denotes positive numbers whose value may change from one line to the next.

3. Definition of the class of L-functions $\mathcal{S}$

We define a class of functions $\mathcal{S}$ as follows. A function F is in $\mathcal{S}$ if

1) Dirichlet Series representation: For $\Re(s) > 1$, $F(s)$ can be represented as an absolutely convergent Dirichlet series $F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}$.
2) Analytic continuation: There exists a non-negative integer m such that $(s-1)^m F(s)$ is an entire function of finite order.
3) Functional equation: $F(s)$ satisfies the functional equation

$$\Xi_F(s) = w_F \overline{\Xi_F(1-\overline{s})},$$

where

$$\Xi_F(s) := F(s) Q^s \prod_{j=1}^{f} \Gamma(\lambda_j s + \mu_j) \tag{3.1}$$

with positive real numbers Q, λ_j and complex numbers w_F, μ_j with $|w_F| = 1, \Re\frac{\mu_j}{\lambda_j} > -\frac{1}{2}$.

4) Euler product: For $\Re(s)$ sufficiently large, $F(s)$ has the Euler product representation

$$F(s) = \prod_p F_p(s),\ F_p(s) = \exp\left(\sum_{k=1}^{\infty} \frac{b_F(p^k)}{p^{ks}}\right),$$

where $b_F(p^k)$ are some coefficients satisfying $b_F(p^k) \ll p^{k\theta_F}$, for some constant $\theta_F < 1/2$.

For $F \in \mathcal{S}$, we define the degree d_F, weight λ, and conductor q_F as

$$d_F = 2\sum_{j=1}^{f} \lambda_j,\ \lambda = \prod_{j=1}^{f} \lambda_j^{2\lambda_j},\ q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{f} \lambda_j^{2\lambda_j}. \tag{3.2}$$

Define the analytic conductor of $F(s)$ as

$$q_F(s) = Q^2 \prod_{j=1}^{f} (|\lambda_j s + \mu_j| + 3)^{2\lambda_j},\ Q_F = q_F(0). \tag{3.3}$$

Let ψ be a Dirichlet character. The twisted L-function F_ψ is defined as

$$F_\psi(s) = \sum_{n=1}^{\infty} \frac{a_F(n)\psi(n)}{n^s}, \text{ for } \Re(s) > 1.$$

We define a subclass of $\mathcal{S}$, denoted by $\mathcal{S}^*$, which consists of L-functions that satisfy the following additional conditions.

(i) $F_\psi \in \mathcal{S}$ for any primitive character $\psi \pmod g$ and $d_{F_\psi} = d_F, Q_{F_\psi} \ll Q_F g^{d_F}$.

(ii) F_ψ is entire for all primitive characters ψ with the exception of at most one primitive character $\psi^* \pmod{g^*}$.

(iii) For any $Q \gg 1$, there exists B_Q which is either 1 or a prime $\gg_F \log_2 Q$, such that

$$1 - \sigma \gg_F \frac{1}{\log(Q(|\Im(s)| + 2))} \tag{3.4}$$

whenever $F_\psi(\sigma + it) = 0$ and $\psi \pmod g$ is a Dirichlet character with square-free conductor $\tilde{g} \le Q$ and $(\tilde{g}, B_Q) = 1$.

(iv) For $\Re(s) > 1$, denote

$$-\frac{F'}{F}(s) := \sum_{n=1}^{\infty} \frac{\Lambda_F(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\Lambda(n)\lambda_F(n)}{n^s}.$$

Then we have

$$|\Lambda_F(n)| \le d_F n^{\theta_F} \log n, \tag{3.5}$$

and as $x \to \infty$, we have

$$\sum_{n \le x} \Lambda(n)|\lambda_F(n)|^2 = x(1 + o(1)). \tag{3.6}$$

(v) For $x \gg_{d_F} (Q_F \nu)^c$ with $c = c(d_F)$ is some constant depending only on d_F, we have

$$\sum_{x \le n \le xe^{1/\nu}} |\Lambda_F(n)| \ll_{d_F} \frac{x}{\nu}. \tag{3.7}$$

From the definition of $\mathcal{S}$, we have the following properties.

Lemma 3.1 (Convexity Bound). *Let $F \in \mathcal{S}$ be as above. Define*

$$\mu_F(\sigma) = \limsup_{|t| \to \infty} \frac{\log |F(\sigma + it)|}{\log |t|}.$$

Then $\mu_F(\sigma)$ is a convex function, and

$$\mu_F(\sigma) \le \begin{cases} 0, & \text{if } \sigma > 1, \\ \frac{1}{2} d_F(1 - \sigma), & \text{if } 0 \le \sigma \le 1, \\ (\frac{1}{2} - \sigma) d_F, & \text{if } \sigma < 0. \end{cases}$$

Proof. This is follows from the general theory of L-functions that can be found in Theorem 6.8 in [45]. □

Lemma 3.2. *Let ψ be a primitive character. Let $F \in \mathcal{S}^*$, then we have*

(1) For $\Re(s) \ge 1$, $\frac{F'_\psi(s)}{F_\psi(s)}$ has no poles except for the character ψ^*, where it has a simple pole at $s = 1$.

(2) For any $\kappa > 1$, we have

$$\sum_{n=1}^{\infty} \frac{|(\Lambda_F * \psi)(n)|}{n^\kappa} \ll \frac{1}{(\kappa - 1)^2}.$$

(3) Let B_Q be defined in condition (iii). *Let ψ be any primitive character with square-free conductor $g \le Q$ such that $(g, B_Q) = 1$. There exists some constant $c = c(F) > 0$ such that for $\Re(s) \ge 1 - c/\log(Q(|\Im s| + 2))$*

$$\frac{F'_\psi(s)}{F_\psi(s)} \ll_F \log^2(Q(|\Im s| + 2)).$$

If $F_\psi(z)$ has no zeros for $\Re(s) > 1 - a$ for some $a > 0$, then the above bound hold for $\Re(s) > 1 - a + 1/\log(Q(|\Im s| + 2))$.

Proof. For (1), it follows from the definition of $\mathcal{S}^*$. For (2), we have

$$\sum_{n=1}^{\infty} |(\Lambda_F * \chi)(n)| n^{-\sigma} \le \sum_{n=1}^{\infty} |\Lambda_F(n)| n^{-\sigma} \sum_{n=1}^{\infty} n^{-\sigma}. \tag{3.8}$$

From (3.6) and partial summation, we have

$$\sum_{n=1}^{\infty} \frac{|\Lambda_F(n)|}{n^\sigma} \ll \sum_{n=1}^{\infty} \frac{\Lambda(n) + \Lambda(n)|\lambda_F(n)|^2}{n^\sigma} \ll \frac{1}{\sigma - 1}, \tag{3.9}$$

which together with (3.8) yields the desired conclusion. To prove part (3), we can choose $c = c(F)$ such that $\Re(\rho_{F_\psi}) \le 1 - \frac{2c}{\log(Q(|\Im(s)|+2))}$ for all non-trivial zeros ρ_{F_ψ} of F_ψ by assumption (iii). Similar to [28, Proposition 5.7 (2)], we have

$$\frac{F'_\psi(s)}{F_\psi(s)} + \frac{m}{s} + \frac{m}{s-1} - \sum_{|s+\mu_{\psi,j}|<1} \frac{1}{s + \frac{\mu_{\psi,j}}{\lambda_{\psi,j}}} - \sum_{|s-\rho_{F_\psi}|<1} \frac{1}{s - \rho_{F_\psi}} \ll \log q_{F_\psi}(s)$$

for some absolute constant. Since $\Re(s) \ge 1 - \frac{c}{\log q_{F_\psi}(s)}$, $\Re(\frac{\mu_{\psi,j}}{\lambda_{\psi,j}}) > -\frac{1}{2}$, and $d_{F_\psi} = d_F$ we have

$$\frac{m}{s} + \frac{m}{s-1} - \sum_{|s+\mu_{\psi,j}|<1} \frac{1}{s + \frac{\mu_{\psi,j}}{\lambda_{\psi,j}}} \ll_F d_F \log q_{F_\psi}(s).$$

Since $Q_{F_\psi} \ll Q_F g^{d_F}$, we have

$$\log q_{F_\psi}(s) \ll \log Q_F g^{d_F} (|\Im s| + 2)^{d_F} \ll_F \log Q(|\Im s| + 2).$$

We also have from [28, Proposition 5.7, (1)] that the number of zeros ρ_{F_ψ} such that $|\Im \rho_{F_\psi} - \Im s| < 1$ is bounded by $\log q_{F_\psi}(s)$. Therefore,

$$\begin{aligned} \frac{F'_\psi(s)}{F_\psi(s)} &\ll d_F \log(q_{F_\psi}(s)) + \sum_{|s-\rho_{F_\psi}|<1} \frac{1}{s - \rho_{F_\psi}} \\ &\ll_F \log^2 Q(|\Im s| + 2), \end{aligned}$$

for all ψ with square-free conductor $g \le Q$ and $(g, B_Q) = 1$. A similar argument can be applied when F_ψ has no zero in the region $\Re(s) > 1 - a$. □

4. Properties of automorphic L-functions

In this section, we will show that L-functions associated to irreducible cuspidal representations of $GL(n)$ belong to the class $\mathcal{S}^*$.

Let π be an irreducible cuspidal automorphic representation of $GL(d_\pi)$ over $\mathbb{Q}$, with unitary central character. For $\Re(s) > 1$, let

$$L(s, \pi) := \sum_{n=1}^{\infty} \frac{a_\pi(n)}{n^s} = \prod_{p \text{ prime}} \prod_{j=1}^{d_\pi} \left(1 - \frac{\alpha_\pi(p, j)}{p^s}\right)^{-1}$$

be the global L-function attached to π (cf. R. Godement and H. Jacquet [25], H. Jacquet and J. Shalika[29, 30]). Denote by $\lambda_\pi(p^k)$,

$$\lambda_\pi(p^k) = \sum_{j=1}^{d_\pi} \alpha_\pi(p,j)^k. \tag{4.1}$$

Then for $\Re(s) > 1$, we have

$$-\frac{L'}{L}(s,\pi) := \sum_{n=1}^{\infty} \frac{\Lambda_\pi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\Lambda(n)\lambda_\pi(n)}{n^s},$$

where $\Lambda(n)$ is the von Mangoldt function. It is known that $L(s,\pi)$ can be analytically continued to an entire function

$$\Phi(s,\pi) = q_\pi^{s/2}\gamma(s,\pi)L(s,\pi), \tag{4.2}$$

which satisfies the functional equation

$$\Phi(s,\pi) = \epsilon_\pi \overline{\Phi}(1-s,\pi),$$

where $\overline{\Phi}(s) = \overline{\Phi(\overline{s})}$ and $\gamma(s,\pi) = \prod_{j=1}^{d_\pi} \Gamma_{\mathbb{R}}(s+\mu_j), \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2), \mu_j \in \mathbb{C}, |\epsilon_\pi| = 1$. We also have the bound

$$|\lambda_\pi(n)| \le d_\pi n^{\theta_\pi}, -\Re(\mu_j) \le \theta_\pi \tag{4.3}$$

for some $\theta_\pi < \frac{1}{2}$. The Generalized Ramanujan Conjecture asserts that $\theta_\pi = 0$. It is known from W. Luo, Z. Rudnick and P. Sarnark [36] that $\theta_\pi \le \frac{1}{2} - \frac{1}{d_\pi^2+1}$. When $K = \mathbb{Q}$ and $d_\pi = 2$, Kim and Sarnak [32] improved the bound to $|\alpha_j(p)| \le \frac{7}{64}$ based on the work of Kim on the symmetric fourth L-functions. V. Blomer and F. Brumley [4] extended this bound to general number fields and obtained better bounds for $GL(3)$ ($c \le \frac{5}{14}$) and $GL(4)$ ($c \le \frac{9}{22}$) L-functions over general number field.

Given a Dirichlet character $\psi \mod g$, where $(g, q_\pi) = 1$, let

$$L(s,\pi\otimes\psi) := \sum_{n=1}^{\infty} \frac{a_\pi(n)\psi(n)}{n^s} = \prod_{p \text{ prime}} \prod_{j=1}^{d_\pi} \left(1 - \frac{\alpha_j(p)\psi(p)}{p^s}\right)^{-1}, \text{ for } \Re(s) > 1.$$

We have

$$-\frac{L'}{L}(s,\pi\otimes\psi) = \sum_{n=1}^{\infty} \frac{\Lambda_\pi(n)\psi(n)}{n^s}, \ \Re(s) > 1.$$

It is known that $L(s,\pi\otimes\psi)$ can be analytically continued to an entire function, and furthermore

$$\Phi(s,\pi\otimes\psi) = (g^{d_\pi} q_\pi)^{s/2}\gamma_\psi(s,\pi)L(s,\pi\otimes\psi)$$

is an entire function of order 1 satisfying the functional equation

$$\Phi(s,\pi\otimes\psi) = \epsilon_{\pi,\psi}\overline{\Phi}(1-s,\pi\otimes\psi),$$

where $\overline{\Phi}(s,\pi\otimes\psi) = \overline{\Phi(\overline{s},\pi\otimes\psi)}, \gamma_\psi(s,\pi) = \prod_{j=1}^{d_\pi} \Gamma_{\mathbb{R}}(s+\mu_{j,\psi}), \Re\mu_{j,\psi} > -\frac{1}{2}, |\epsilon_{\pi,\psi}| = 1$. This shows that $GL(n)$ L-functions belong to $\mathcal{S}$ and it remains to prove conditions (i)-(v).

Condition (i) can be verified by properties of Rankin-Selberg L-functions (see [28, Section 5, p. 97, eq (5.11)]). Condition (ii) is satisfied for $GL(1)$ L-functions and when $d_\pi \ge 2$, we know that $L(s,\pi\otimes\psi)$ is entire and thus condition (ii) is also satisfied. The zero free region of $L(s,\pi\otimes\psi)$ can be found in [5, Proposition 2.11]. In particular, there is only possible one real zero in the region $\sigma > 1 - \frac{c}{\log(Q_\pi(|t|+2))}$ for any irreducible cuspidal automorphic representation π of $GL(d_\pi)$, where c is an absolute positive constant depending only on d_π. The exceptional zero is called a Siegel zero. It is believed that the only possibility of a Siegel zero is from a Dirichlet L-function associated to a quadratic character. In fact, J. Hoffstein and D. Ramakrishnan proved that there is no Siegel zero for cups forms on $GL(n)$ for $n > 1$ if the functoriality of Langlands holds. This implies that cusp forms on $GL(2)$ admit no Siegel zeros. W. Banks [3] proved the non-existence of Siegel zeros for cups forms on $GL(3)$. Thus condition (iii) in the definition of $\mathcal{S}^*$ is satisfied for $d_\pi = 2$ or 3. For $GL(1)$ L-functions, we know (iii) is true from Theorem 4.1 below. For $GL(n)$ L-functions, we prove an analogue in Theorem 4.2.

Theorem 4.1 (Landau-Page,[21, Corollary 6])**.** *For $Q \geq 100$, there exists B_Q which is either 1 or a prime $\gg \log_2 Q$ such that $1-\sigma \gg \frac{1}{\log Q(|t|+1)}$ whenever $L(\sigma+it,\chi)=0$ and χ is a Dirichlet character modulo q with $q \leq Q, (q, B_Q) = 1$.*

Theorem 4.2. *Let π be an irreducible cuspidal representation of $GL(n)$, and let Q be a sufficiently large integer. Then, there exists a quantity B_Q which is either 1 or a prime of size $\gg \log_2 Q$ such that $L(s, \pi\otimes\psi)$ has no zero in the region*

$$1-\Re(s) \ll \frac{1}{\log Q(|t|+1)},$$

whenever the conductor of ψ is squarefree and coprime to B_Q. All implied constants only depend on π.

Lemma 4.3 (J. Hoffstein and D. Ramakrishnan, [27, Theorem A]; [5, Remark 2.12])**.** *Let π be an irreducible cuspidal automorphic representation of $GL(n)$ with $Q_\pi \leq Q$. Then there is an absolute constant $c>0$ such that $L(s,\pi)$ has no zeros in the interval $1-\frac{c}{\log Q} \leq \sigma \leq 1$ with the exception of at most one of such π.*

Lemma 4.4 (F. Brumley, [13, Corollary 6])**.** *Let π and π' be cuspidal automorphic representations of $GL_n(\mathbb{A})$ with analytic conductor $\leq Q$ and $t \in \mathbb{R}$. There exist constants $c = c(n,n') > 0$ and $A = A(n,n') > 0$ such that $L(\sigma, \pi\times\pi')$ has no zeros in the interval*

$$1-\frac{c}{Q^A} \leq \sigma \leq 1.$$

Proof. [Proof of Theorem 4.2] Let ψ be a Dirichlet character of squarefree conductor g and π be a cusp form on $GL(n)$ with conductor Q_π. Then $\pi\otimes\psi$ is a cusp form on $GL(n)$ with conductor $\ll Q_\pi g^n$. From [5, Proposition 2.11], we have $L(s,\pi\otimes\psi)$ has no zeros for $\Re(s) \geq_\pi 1-\frac{c}{\log Q(|t|+1)}$ for all $g \leq Q$, with exception of at most one real zero. From Lemma 4.3, we see that for all primitive characters with conductor at most Q, there is at most one exceptional character $\psi_Q \pmod{g_Q}$ such that it has a real zero β satisfying

$$1-\beta \ll_\pi \frac{1}{\log Q}.$$

From Lemma 4.4, we see that

$$1-\beta \gg_\pi \frac{1}{(Q_\pi g_Q^n)^A}.$$

Thus, $g_Q \gg_\pi (\log Q)^{1/A'}$. Since g_Q is squarefree, by prime number theorem, there exists $B_Q \gg \log g_Q \gg \log_2 Q$ such that $L(s,\pi\otimes\psi)$ has no zero in the region

$$\Re(s) \geq 1-\frac{c}{\log Q},$$

for all Dirichlet characters ψ with squarefree conductor at most Q and coprime to B_Q. □

Condition (iv) follows from Rankin-Selberg theory. A proof can be found in [28, Theorem 5.13] (see also [35, Lemma 5.2]). Condition (v) is also satisfied for automorphic L-functions of $GL(n)$ (See [44, eq (1.10)]). Therefore, we see that L-functions associated to irreducible automorphic representations of $GL(n)$ belong to $\mathcal{S}^*$.

5. Moment of the Resonator

Theorem 5.1. *Let $F \in \mathcal{S}$. Suppose $X(s) = \sum_{n\le M} \frac{x_n}{n^s}, Y(s) = \sum_{n\le M} \frac{y_n}{n^s}$ with $x_n = \overline{y_n}$, and $M \le T$. Then we have*

$$S_0 = \left(\frac{1}{2\pi}\int_{T_1}^{T_2} \log(\lambda Q^2 t^{d_F})dt\right)\sum_{m\le M}\frac{x_m y_m}{m} - \frac{T_2-T_1}{2\pi}\sum_{m\le M}\frac{(\Lambda_F * x)(m)y_m + \overline{\Lambda_F} * y(m)x_m}{m} + \mathcal{E}_0,$$

where

$$\mathcal{E}_0 = O\left((\log T)^2\left(M\left\|\frac{x_n}{n}\right\|_1\left\|\frac{y_n}{n}\right\|_1 + M^{\theta_F+\epsilon}\|x_n\|_1\left\|\frac{y_n}{n}\right\|_1 + M^{\theta_F+\epsilon}\|y_n\|_1\left\|\frac{x_n}{n}\right\|_1\right)\right) + O\left((\log T)^2 M^{1+\theta_F+\epsilon}\left(\left\|\frac{x_n}{n}\right\|_1\|y_n\|_\infty + \left\|\frac{y_n}{n}\right\|_1\|x_n\|_\infty\right)\right). \tag{5.1}$$

Proof. From the residue theorem, we have for any $c > 1$,

$$S_0 = \frac{1}{2\pi i}\left(\int_{c+iT_1}^{c+iT_2} + \int_{c+iT_2}^{1-c+iT_2} + \int_{1-c+iT_1}^{c+iT_1} + \int_{1-c+iT_2}^{1-c+iT_1}\right)X(s)Y(1-s)\frac{F'}{F}(s)ds = J_R - J_L + J_H, \tag{5.2}$$

where

$$J_R = \frac{1}{2\pi i}\int_{c+iT_1}^{c+iT_2} X(s)Y(1-s)\frac{F'}{F}(s)ds, \tag{5.3}$$

$$J_L = \frac{1}{2\pi i}\int_{1-c+iT_1}^{1-c+iT_2} X(s)Y(1-s)\frac{F'}{F}(s)ds, \tag{5.4}$$

$$J_H = \frac{1}{2\pi i}\left(\int_{c+iT_2}^{1-c+iT_2} + \int_{1-c+iT_1}^{c+iT_1}\right)X(s)Y(1-s)\frac{F'}{F}(s)ds. \tag{5.5}$$

For J_H, we first note that

$$|X(s)Y(1-s)| = \left|\sum_{u\le M}\frac{x_u}{u^s}\sum_{k\le M}\frac{y_k}{k^{1-s}}\right| \le M\left\|\frac{x_n}{n}\right\|_1\left\|\frac{y_n}{n}\right\|_1 + M^{c-1}\|x_n\|_1\left\|\frac{y_n}{n}\right\|_1 + M^{c-1}\|y_n\|_1\left\|\frac{x_n}{n}\right\|_1, \tag{5.6}$$

where each part corresponds to a bound for $0 \le \Re(s) \le 1$, $1-c \le \Re(s) \le 0$, and $1 \le \Re(s) \le c$ respectively. Since $T_1 = T + O(1)$ and $T_2 = 2T + O(1)$ are chosen such that

$$\frac{F'}{F}(\sigma + iT_1) \ll (\log T)^2,\ \frac{F'}{F}(\sigma + iT_2) \ll (\log T)^2, \tag{5.7}$$

uniformly for $\sigma \in [-1, 2]$, it follows that

$$J_H \ll (\log T)^2\left(M\left\|\frac{x_n}{n}\right\|_1\left\|\frac{y_n}{n}\right\|_1 + M^{c-1}\|x_n\|_1\left\|\frac{y_n}{n}\right\|_1 + M^{c-1}\|y_n\|_1\left\|\frac{x_n}{n}\right\|_1\right). \tag{5.8}$$

Taking logarithmic derivative of the functional equation (3.1), we have

$$\frac{F'}{F}(s) = \frac{\Delta_F'}{\Delta_F}(s) - \frac{\overline{F'}}{\overline{F}}(1-s)$$

where $\Delta_L(s) = wQ^{1-2s}\prod_{j=1}^{f}\frac{\Gamma(\lambda_j(1-s)+\overline{\mu_j})}{\Gamma(\lambda_j s+\mu_j)}$. Therefore,

$$
\begin{aligned}
J_L &= \frac{1}{2\pi i}\int_{1-c+T_1}^{1-c+iT_2} X(s)Y(1-s)\frac{F'}{F}(s)ds \\
&= \overline{\frac{1}{2\pi i}\int_{c+iT_1}^{c+iT_2}\left\{\frac{\Delta_F'}{\Delta_F}(s)-\frac{F'}{F}(s)\right\}\overline{X}(1-s)\overline{Y}(s)ds}. \qquad (5.9)
\end{aligned}
$$

We write

$$J_L = K - \overline{I_R}, \qquad (5.10)$$

where

$$K = \overline{\frac{1}{2\pi i}\int_{c+iT_1}^{c+iT_2}\frac{\Delta_F'}{\Delta_F}(s)\overline{Y}(s)\overline{X}(1-s)ds}$$

and

$$I_R = \frac{1}{2\pi i}\int_{c+iT_1}^{c+iT_2}\frac{F'}{F}(s)\overline{Y}(s)\overline{X}(1-s)ds.$$

If $X(s)=\overline{Y}(s)$, then we have

$$I_R = J_R. \qquad (5.11)$$

From Stirling's formula, we have

$$\frac{\Delta_F'}{\Delta_F}(s) = -\log\left(\lambda Q^2|t|^{d_F}\right) + O\left(\frac{1}{|t|}\right), \qquad (5.12)$$

and thus by (5.6),

$$
\begin{aligned}
K = &-\frac{1}{2\pi}\int_{T_1}^{T_2}\log\left(\lambda Q^2|t|^{d_F}\right)Y(c-it)X(1-c+it)dt \\
&+ O\left(\log T\left(M\left\|\frac{x_n}{n}\right\|_1\left\|\frac{y_n}{n}\right\|_1 + M^{c-1}\|x_n\|_1\left\|\frac{y_n}{n}\right\|_1 + M^{c-1}\|y_n\|_1\left\|\frac{x_n}{n}\right\|_1\right)\right). \qquad (5.13)
\end{aligned}
$$

The main term in K, denoted by K_0, is given by

$$
\begin{aligned}
K_0 &= -\frac{1}{2\pi}\int_{T_1}^{T_2}\log(\lambda Q^2 t^{d_F})\sum_{u\le M}\frac{x_u}{u^{1-c+it}}\sum_{k\le M}\frac{y_k}{k^{c-it}}dt \\
&= -\frac{1}{2\pi}\sum_{u\le M}\frac{x_u}{u^{1-c}}\sum_{k\le M}\frac{y_k}{k^c}\int_{T_1}^{T_2}\log\left(\lambda Q^2 t^{d_F}\right)\left(\frac{k}{u}\right)^{it}dt \\
&= K_d + K_{nd}, \qquad (5.14)
\end{aligned}
$$

where K_d denotes the contribution from the diagonal terms with $k=u$, and K_{nd} denotes the contribution from the off-diagonal terms with $k\neq u$. We have

$$
\begin{aligned}
K_d &= -\frac{1}{2\pi}\sum_{u\le M}\frac{x_u y_u}{u}\int_{T_1}^{T_2}\log\left(\lambda Q^2 t^{d_F}\right)dt \\
&= -\left(\frac{d_F}{2\pi}T\log T + O(T)\right)\sum_{u\le M}\frac{x_u y_u}{u}. \qquad (5.15)
\end{aligned}
$$

For K_{nd}, we have

$$K_{nd} \ll \sum_{\substack{u,k\le M\\ u\ne k}} \frac{|x_u y_k|}{u^{1-c}k^c}\frac{\log T}{|\log k/u|}$$
$$\ll \log T M^{c-1}\sum_{u\le M}|x_u|\sum_{k\le M}\frac{|y_k|}{k^c} + \log T\sum_{u\le M}|x_u|\sum_{u/2\le k\le 2u}\frac{|y_k|}{k^c}\frac{u}{|k-u|}$$
$$\ll \log T M^{c-1}\|x_n\|_1\left\|\frac{y_k}{k^c}\right\|_1 + \log T\|x_n\|_1\|y_n\|_\infty \log M. \tag{5.16}$$

For J_R, we have

$$J_R = \frac{1}{2\pi i}\int_{c+iT_1}^{c+iT_2} X(s)Y(1-s)\frac{F'}{F}(s)ds$$
$$= -\frac{1}{2\pi}\int_{T_1}^{T_2}\sum_{u\le M}\frac{x_u}{u^{c+it}}\sum_{k\le M}\frac{y_k}{k^{1-c-it}}\sum_{n=1}^{\infty}\frac{\Lambda_F(n)}{n^{c+it}}dt$$
$$= J_d + J_{nd}, \tag{5.17}$$

where J_d denotes the contribution from the diagonal terms with $k = nu$, and J_{nd} denotes the contribution from the off diagonal terms with $k \ne nu$.

$$J_d = -\frac{T_2 - T_1}{2\pi}\sum_{n=1}^{\infty}\sum_{u\le M}\frac{\Lambda_F(n)x_u y_{nu}}{nu}, \tag{5.18}$$

and for J_{nd} we have

$$J_{nd} \ll \log T\sum_{n=1}^{\infty}\frac{|\Lambda_F(n)|}{n^c}\sum_{u\le M}\frac{|x_u|}{u^c}\sum_{\substack{k\le M\\ k\ne nu}}\frac{|y_k|}{k^{1-c}}\frac{1}{\log|k/nu|}$$
$$\ll \log T\sum_{n=1}^{\infty}\frac{|\Lambda_F(n)|}{n^c}\sum_{u\le M}\frac{|x_u|}{u^c}M^{c-1}\max_{h\ne k}\sum_{k\le M}\frac{|y_k|}{|\log(k/h)|}$$
$$\ll \log T M^{c-1}\sum_{n=1}^{\infty}\frac{|\Lambda_F(n)|}{n^c}\left\|\frac{x_n}{n^c}\right\|_1\left(\|y_n\|_1 + \|y_n\|_\infty M\log M\right). \tag{5.19}$$

Taking $c = 1 + \theta_F + \epsilon$, and combining (5.8), (5.13), (5.15), (5.16), (5.17), (5.18), and (5.19), we complete the proof. □

6. First moment

In this section, we obtain asymptotic formulas for S_1 defined in (2.1) for $F \in \mathcal{S}^*$ in Theorem 6.1. We will see that Theorem 6.1 is a consequence of (6.11), Theorem 6.3 and Theorem 6.4. We first prove (6.11), then we prove Theorem 6.3 and Theorem 6.4.

Theorem 6.1. *Let $F \in \mathcal{S}^*$, ψ^* (mod g^*) be as in* (ii)*. Let χ be a primitive character modulo q. If x_n, y_n are multiplicative and supported on squarefree integers up to M whose prime factors are coprime to B_M (defined in* (iii)*) and are congruent to* 1 *modulo* $\mathrm{lcm}(q, g^*)$*. Then there exists some* *constant* $c = c(F) > 0$ *such that uniformly for* $M \le \exp(\sqrt{\log T})$,

$$
\begin{aligned}
S_1 =& \frac{T}{2\pi}\left(\sum_{nu\le M}\frac{x_u y_{nu}}{nu}r_0(n) - \sum_{sv\le M}\frac{x_s y_{sv}}{sv}r_1(v)\right)\\
&- \frac{T}{2\pi}\sum_{u\le M}\sum_{\substack{v\le M\\(v,u)=1}}\frac{x_u y_v r_3(u)}{uv}\sum_{s\le M}\frac{y_s x_s}{s}\\
&+ O\left(T^{\theta_F+\epsilon}\|x_n\|_\infty\left\|\frac{y_n}{n}\right\|_1 + T^{\epsilon}\left\|\frac{x_n}{n}\right\|_1(\|y_n\|_\infty M + \|y_n\|_1)\right)\\
&+ O\left(q^{1/2}T^{1/2}\mathcal{L}^3\|x_n\|_1\left\|\frac{\overline{y}_n}{n}\right\|_1 + q^{1/2+\theta_F+\epsilon}T^{1/2+\theta_F+\epsilon}M^{\theta_F+\epsilon}\|y_n\|_\infty\|x_n\|_1\right)\\
&+ \mathcal{E} + \mathcal{E}',
\end{aligned}
$$

where

$$
\begin{aligned}
r_0(n) &= d_F P_1\left(\log\left((\lambda Q^2)^{1/d_F}T\right)\right) - (\Lambda_F * 1)(n),\\
r_1(v) &= \log\left(\frac{2qT}{\pi ve}\right)\overline{f_{-1}} + \overline{\frac{\tau(\overline{\chi})\tau(\psi^*)}{\phi(q)}\mu(q/\ell_0)\overline{\psi^*}(q/\ell_0)L(1,\chi\psi^*)\tilde{f}_{-1}},\ \ \ell_0 = \gcd(q, g^*),\\
G(z,\overline{\chi}) &= \sum_{d=1}^{\infty}\frac{\Lambda_F(d)\overline{\chi}(d)}{d^z} = \frac{f_{-1}}{z-1} + f_0 + f_1(z-1) + \dots,\\
G(z,\psi^*) &= \sum_{d=1}^{\infty}\frac{\Lambda_F(d)\psi^*(d)}{d^z} = \frac{\tilde{f}_{-1}}{z-1} + \tilde{f}_0 + \tilde{f}_1(z-1) + \dots,\\
r_3(u) &= -\Lambda(u)\overline{f_{-1}} + \overline{r_4(u)},\\
r_4(u) &= \sum_{hk=u}\mu(k)\left(\widetilde{X_1}(h,kq) + f_{-1}\widetilde{X_2}(k)\right),\\
\widetilde{X_1}(h,k) &= \sum_{a|(h,k)}\Lambda_F(a) + \sum_{p|h,p\nmid k}\sum_{r=1}^{\infty}\frac{\Lambda_F(p^r)}{p^r}(p-1),\\
\widetilde{X_2}(k) &= f_0 - \eta(1;kq,1) + \Big(\gamma + \sum_{p|kq}\frac{\log p}{p-1}\Big)f_{-1},\\
\eta(1;k,1) &= \sum_{p|k}\sum_{m=1}^{\infty}\frac{\Lambda_F(p^m)}{p^m},\\
\mathcal{E} &\ll M^{\frac{1}{2}+\theta_F+\epsilon}q^{1+\theta_F+\epsilon}T\|x_n\|_1\|y_n\|_\infty\left\|\frac{\tau_3 * |y|(n)}{n}\right\|_1\left\|\frac{(\tau * |y|)(n)}{n}\right\|_1\left\|\frac{y_n}{n}\right\|_1\exp(-c\sqrt{\log T}),\\
\mathcal{E}' &\ll q^{\theta_F+\epsilon}M^{\theta_F+\epsilon}T\left\|\frac{|x_s y_s|}{s}\right\|_1\left\|\frac{(\tau_3 * |x|)(n)}{n}\right\|_1\left\|\frac{y_v}{v}\right\|_1\exp(-c\sqrt{\log T}).
\end{aligned}
$$

If there exist some positive constant $a = a(\chi, F)$ *such that both* F_ψ *and* $L(s,\chi\psi)$ *have no zeros in the region* $\Re(s) \ge 1 - a$ *for all* ψ, *then the term* $\exp(-c\sqrt{\log T})$ *in the error terms* $\mathcal{E}, \mathcal{E}'$ *can be replaced by* $T^{-\delta+\epsilon}$ *for some small enough* $\delta = \delta(F, a) > 0$ *uniformly for* $M \le \sqrt{T}$.

6.1. **Set up.** First we recall the functional equations and definitions of $F(s)$ and $L(s,\chi)$,

$$F(s)=\Delta_F(s)\overline{F}(1-s),\ \overline{F}(s)=\overline{F(\overline{s})}, \tag{6.1}$$

$$\Delta_F(s)=wQ^{1-2s}\prod_{j=1}^{f}\frac{\Gamma(\lambda_j(1-s)+\overline{\mu_j})}{\Gamma(\lambda_j s+\mu_j)}, \tag{6.2}$$

$$L(s,\chi)=B(s)L(1-s,\overline{\chi}), \tag{6.3}$$

$$B(s)=\frac{\tau(\chi)}{i^{\mathfrak{a}}q^{\frac{1}{2}}}\left(\frac{q}{\pi}\right)^{\frac{1}{2}(1-2s)}\frac{\Gamma\left(\frac{1}{2}(1-s+\mathfrak{a})\right)}{\Gamma\left(\frac{1}{2}(s+\mathfrak{a})\right)}, \tag{6.4}$$

where

$$\mathfrak{a}=\begin{cases}0, & \text{if } \chi(-1)=1,\\ 1, & \text{if } \chi(-1)=-1,\end{cases}\quad, B(s)\overline{B(1-s)}=1,$$

and

$$\tau(\chi)=\sum_{m=1}^{q}\chi(m)e_q(m),\quad \tau(\overline{\chi})=\chi(-1)\overline{\tau(\chi)}.$$

From the definition of S_1 in (2.1), the functional equation (6.3) and the residue theorem, we have

$$\begin{aligned}S_1&=\sum_{\substack{F(\rho)=0\\ T_1\le\Im\rho\le T_2}}L(\rho,\chi)X(\rho)Y(1-\rho)\\
&=\sum_{\substack{\overline{F}(1-\rho)=0\\ T_1\le\Im\rho\le T_2}}B(\rho)L(1-\rho,\overline{\chi})X(\rho)Y(1-\rho) &(6.5)\\
&=-\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\overline{F}'}{\overline{F}}(1-s)B(s)L(1-s,\overline{\chi})X(s)Y(1-s)ds &(6.6)\\
&:=-S_R+S_L-S_H,\end{aligned}$$

where $\mathcal{C}$ is the positively oriented rectangle with vertices at $1-\kappa+iT_1,\kappa+iT_1,1-\kappa+iT_2,\kappa+iT_2$, with $\kappa=1+O(\mathcal{L}^{-1}),\mathcal{L}=\log\left(\lambda Q^2T^{d_F}\right)$, $T_1=T+O(1)$ and $T_2=2T+O(1)$ are chosen so that the nearest zeros of $F(s)$ are $\gg\frac{1}{\log T}$ distance away, and S_R,S_L and S_H are defined as

$$S_R=\int_{\kappa+iT_1}^{\kappa+iT_2}\frac{\overline{F}'}{\overline{F}}(1-s)B(s)L(1-s,\overline{\chi})X(s)Y(1-s)ds, \tag{6.7}$$

$$S_L=\int_{1-\kappa+iT_1}^{1-\kappa+iT_2}\frac{\overline{F}'}{\overline{F}}(1-s)B(s)L(1-s,\overline{\chi})X(s)Y(1-s)ds, \tag{6.8}$$

$$\begin{aligned}S_H=&\int_{1-\kappa+iT_1}^{\kappa+iT_1}\frac{\overline{F}'}{\overline{F}}(1-s)B(s)L(1-s,\overline{\chi})X(s)Y(1-s)ds\\
&-\int_{1-\kappa+iT_2}^{\kappa+iT_2}\frac{\overline{F}'}{\overline{F}}(1-s)B(s)L(1-s,\overline{\chi})X(s)Y(1-s)ds. &(6.9)\end{aligned}$$

By Stirling's formula, for $t>0$, equation (6.2) becomes

$$\Delta_F(s)=\left(\lambda Q^2t^{d_F}\right)^{\frac{1}{2}-\sigma-it}\exp\left(itd_F+\frac{i\pi(\mu-d_F)}{4}\right)\left(w+O\left(\frac{1}{t}\right)\right), \tag{6.10}$$

where

$$\mu=2\sum_{j=1}^{f}(1-2\Re\mu_j),\quad \lambda=\prod_{j=1}^{f}\lambda_j^{2\lambda_j}.$$

6.2. **Horizontal Integral.** From Lemma 3.1, Lemma 3.2 and (6.10), we have the bounds

$$\begin{aligned}
&\frac{F'}{F}(\sigma+it) \ll_F \log^2 t,\\
&B(\sigma+it) \ll t^{\frac{1}{2}-\sigma},\\
&L(\sigma+it,\chi) \ll \frac{t^{\frac{1-\sigma}{2}}}{1-\sigma}, 1/2 \le \sigma \le 1-\frac{A}{\log T},\\
&L(\sigma+it,\chi) \ll \log t, \sigma > 1-\frac{A}{\log t},\\
&L(\sigma+it,\chi) = B(\sigma)L(1-\sigma-it,\chi) \ll t^{1/2}\log t, -\frac{A}{\log t} < \sigma < 0,\\
&X(s) \ll M^{1-\sigma}\left\|\frac{x_n}{n}\right\|_1, 1-\kappa \le \sigma \le \kappa,\\
&Y(1-s) \ll M^{\sigma}\left\|\frac{y_n}{n}\right\|_1, 1-\kappa \le \sigma \le \kappa.
\end{aligned}$$

It follows that the horizontal integral S_H (6.9) is bounded by

$$MT^{\frac{1}{2}}\left\|\frac{x_n}{n}\right\|_1\left\|\frac{y_n}{n}\right\|_1 \log^3 T. \tag{6.11}$$

6.3. **Right integral.** By taking the logarithmic derivative of the functional equation of $F(s)$, we find that

$$\frac{F'}{F}(s) = \frac{\Delta_F'}{\Delta_F}(s) - \frac{\overline{F}'}{\overline{F}}(1-s),$$

and so the right integral S_R defined in (6.7) becomes

$$S_R = \frac{1}{2\pi i}\int_{\kappa+iT_1}^{\kappa+iT_2}\left\{\frac{\Delta_F'}{\Delta_F}(s) - \frac{F'}{F}(s)\right\} L(s,\chi)X(s)Y(1-s)ds. \tag{6.12}$$

Next, we use the following lemma to evaluate S_R.

Lemma 6.2. *Set $D(s) := \sum_{n=1}^{\infty}\alpha_n n^{-s}$. Suppose that there exists $\alpha > 0$ such that $\sum_{n=1}^{\infty}|\alpha_n|n^{-\sigma} \ll (\sigma-1)^{-\alpha}$ as $\sigma \to 1$. Suppose that $|\alpha_n| \ll n^{\theta_D+\epsilon}$. Then for $M \le T$, we have*

$$\begin{aligned}
J_k(T) :&= \frac{1}{2\pi i}\int_{\kappa+iT_1}^{\kappa+iT_2}\left(\frac{\Delta_F'(s)}{\Delta_F(s)}\right)^k D(s)X(s)Y(1-s)ds\\
&= \frac{(-1)^k d_F^k T P_k(\log((\lambda Q^2)^{\frac{1}{d_F}}T))}{2\pi}\sum_{nu\le M}\frac{\alpha_n x_u y_{nu}}{nu}\\
&\quad + O_k\left(T^{\theta_D+\epsilon}\|x_n\|_\infty\left\|\frac{y_n}{n}\right\|_1 + T^{\epsilon}\left\|\frac{x_n}{n}\right\|_1(\|y_n\|_\infty M + \|y_n\|_1)\right).
\end{aligned}$$

Proof. From (6.10), we have

$$\frac{\Delta_F'}{\Delta_F}(s) = -\log\left(\lambda Q^2 t^{d_F}\right) + O\left(\frac{1}{t}\right),$$

for $1/2 \le \sigma \le 2$ and $t \ge 1$, $\kappa = 1+1/\mathcal{L}$ and $\mathcal{L} = \log(\lambda Q^2 T^{d_F})$. Thus,

$$J_k(T) = \frac{1}{2\pi i}\int_{\kappa+iT_1}^{\kappa+iT_2}\left((-\log(\lambda Q^2 t^{d_F}))^k + O_k(\mathcal{L}^{k-1}t^{-1})\right)D(s)X(s)Y(1-s)ds.$$

The error terms contribute at most

$$\frac{1}{T}\mathcal{L}^{k-1}\int_{T_1}^{T_2}\sum_{n=1}^{\infty}\frac{|\alpha_n|}{n^\kappa}\sum_{u=1}^{\infty}\frac{|x_u|}{u}\sum_{v=1}^{\infty}|y_v|dt \ll \mathcal{L}^{\alpha+k}\left\|\frac{x_n}{n}\right\|_1\|y_n\|_1.$$

Changing the order of summation and integration gives

$$J_k(T) = \sum_{n,u,v} \frac{\alpha_n x_u y_v (-1)^k}{n^\kappa u^\kappa v^{1-\kappa} 2\pi} \int_{T_1}^{T_2} (\log(\lambda Q^2 t^{d_F}))^k \left(\frac{v}{nu}\right)^{it} dt + O_k\left(T^\epsilon \left\|\frac{x_n}{n}\right\|_1 \|y_n\|_1\right)$$
$$:= J_d + J_{nd} + O_k\left(T^\epsilon \left\|\frac{x_n}{n}\right\|_1 \|y_n\|_1\right),$$

where $s = \kappa + it$, J_d consists of the diagonal terms with $v = nu$, and J_{nd} consists of the off-diagonal terms with $v \neq nu$. Note that

$$\int_{T_1}^{T_2} \log^k(\lambda Q^2 t^{d_F}) dt = d_F^k T P_k(\log((\lambda Q^2)^{1/d_F} T)) + O_k(T^\epsilon),$$

where P_k is a monic polynomial of degree k. Since $|\alpha_n| \ll T^{\theta_D+\epsilon}$, we see that

$$J_d = \frac{(-1)^k}{2\pi} \sum_{nu \le M} \frac{\alpha_n x_u y_{nu}}{nu} \int_T^{2T} \log^k(\lambda Q^2 t^{d_F}) dt$$
$$= \frac{(-1)^k d_F^k T P_k(\log((\lambda Q^2)^{\frac{1}{d_F}} T))}{2\pi} \sum_{nu \le M} \frac{\alpha_n x_u y_{nu}}{nu} + O_k\left(T^{\theta_D+\epsilon} \|x_n\|_\infty \left\|\frac{y_n}{n}\right\|_1\right).$$

For the off-diagonal terms,

$$J_{nd} = \sum_{n,u,v,v\neq nu} \frac{(-1)^k \alpha_n x_u y_v}{n^\kappa u^\kappa v^{1-\kappa} 2\pi} \int_T^{2T} \log^k(\lambda Q^2 t^{d_F}) \left(\frac{v}{nu}\right)^{it} dt$$
$$\ll \mathcal{L}^{k+\alpha} \left\|\frac{x_n}{n}\right\|_1 \sum_{v \le M, v \neq nu} \frac{|y_v|}{v^{1-\kappa} |\log(v/nu)|}.$$

Thus it is enough to consider

$$\max_h \sum_{v \le M, v \neq h} \frac{|y_v|}{v^{1-\kappa} |\log(v/h)|}. \tag{6.13}$$

For $h \geq 2M$, we see that (6.13) can be bounded by $\|y_n\|$ since $\kappa = 1 + O(\mathcal{L}^{-1})$. For $h \leq 2M$, we have

$$\sum_{v \le M, v \neq h} \frac{|y_v|}{|\log(v/h)|} \ll \sum_{\substack{v \le M \\ |v/h| > 3/2 \\ or\ |v/h| < 1/2}} |y_v| + \sum_{\substack{v \le M \\ 1/2 \le |v/h| \le 3/2}} \frac{|y_v|}{|\log(v/h)|}$$
$$\ll ||y_n||_1 + ||y_n||_\infty \sum_{s \le h/2} \left(\frac{1}{|\log(h/(h-s))|} + \frac{1}{|\log(h/(h+s))|}\right)$$
$$\ll ||y_n||_1 + ||y_n||_\infty \sum_{s \le v/2} \frac{h}{s}$$
$$\ll ||y_n||_1 + ||y_n||_\infty M \log M.$$

This gives

$$J_{nd} \ll \mathcal{L}^{k+\alpha} \left\|\frac{x_n}{n}\right\|_1 (\|y_n\|_1 + \|y_n\|_\infty M \log M),$$

which completes the proof. □

Theorem 6.3. *Let $F \in \mathcal{S}^*$. If x_n and y_n are coefficients supported on integers $n \le M \le T$, then the right integral S_R defined in* (6.7) *becomes*

$$S_R = \frac{T}{2\pi} \sum_{nu \le M} \frac{\chi(n) x_u y_{nu}}{nu} \left(-d_F P_1 \left(\log \left((\lambda Q^2)^{1/d_F} T \right) \right) + \sum_{d|n} \Lambda_F(d) \overline{\chi}(d) \right) + O\left(T^{\theta_F + \epsilon} \|x_n\|_\infty \left\| \frac{y_n}{n} \right\|_1 + T^\epsilon \left\| \frac{x_n}{n} \right\|_1 (\|y_n\|_\infty M + \|y_n\|_1) \right).$$

Proof. Apply Lemma 6.2 with $k = 1, \alpha = \chi$ and $k = 0, \alpha = \Lambda_F * \chi$. The assumptions in Lemma 6.2 can be verified from Lemma 3.2. □

6.4. **Left integral.** In this section, we prove the following Theorem.

Theorem 6.4. *Let $F, \psi^* \pmod{g^*}$ and $\chi \pmod q$ be as before. If x, y are multiplicative functions supported on squarefree integers up to M whose prime factors are coprime to B_M and are congruent to 1 modulo $\operatorname{lcm}(q, g^*)$. Then uniformly for $M \le \exp(\sqrt{\log T})$,*

$$\begin{aligned} S_L = &- \frac{T}{2\pi} \sum_{su \le M} \frac{x_{su}}{su} \sum_{\substack{sv \le M \\ (v,uq)=1}} \frac{y_{sv}}{v} \left(\delta(u) \log\left(\frac{2qT}{\pi v e} \right) \overline{f_{-1}} - \Lambda(u) \overline{f_{-1}} \right) \\ &- \frac{T}{2\pi} \sum_{su \le M} \frac{x_{su}}{su} \sum_{\substack{sv \le M \\ (v,uq)=1}} \frac{y_{sv}}{v} \sum_{hk=u} \mu(k) \left(\overline{\widetilde{X_1}(h,k)} + \overline{f_{-1} \widetilde{X_2}(k)} \right) \\ &- \frac{\overline{\tau(\overline{\chi})} \tau(\psi^*) T}{2\pi} \overline{L(1, \chi\psi^*)} \tilde{f}_{-1} \frac{\mu(q/\ell_0) \psi(q/\ell_0)}{\phi(q)} \sum_{s \le M} \frac{x_s}{s} \sum_{\substack{sv \le M \\ (v,uq)=1}} \frac{y_{sv}}{v} \\ &+ O\left(q^{1/2} T^{1/2} \mathcal{L}^3 \|x_n\|_1 \left\| \frac{\overline{y}_n}{n} \right\|_1 + q^{1/2+\theta_F+\epsilon} T^{1/2+\theta_F+\epsilon} M^{\theta_F+\epsilon} \|y_n\|_\infty \|x_n\|_1 \right) \\ &+ \mathcal{E} + \mathcal{E}', \end{aligned}$$

where

$$\ell_0 = \gcd(q, g^*), \;\; g^* = \ell_0 g_0,$$

$$\delta(u) = \begin{cases} 1, & \text{if } u = 1, \\ 0, & \text{if } u > 1, \end{cases}$$

$$G(z, \psi) = \sum_{d=1}^\infty \frac{\Lambda_F(d) \psi(d)}{d^z}, \quad f_{-1} = \operatorname{Res}_{z=1} G(z, \overline{\chi}), \quad f_0 = \lim_{z \to 1} G(z, \overline{\chi}) - \frac{f_{-1}}{z-1},$$

$$\eta(z; p; l, \psi) = \sum_{m=0}^\infty \frac{\Lambda_F(p^{l+m}) \psi(p^m)}{p^{mz}}, \quad \eta(z; k, \psi) = \sum_{p|k} \eta(z; p, 0, \psi),$$

$$\widetilde{f}_{-1} = \operatorname{Res}_{z=1} G(z, \psi^*),$$

$$\widetilde{X_1}(h,k) = \sum_{a|(h,k)} \Lambda_F(a) \overline{\chi}(a) + \sum_{p^l | h, p \nmid k} \overline{\chi}(p^l) \eta(1; p, l, \overline{\chi})(1 - p^{-1}),$$

$$\widetilde{X_2}(k) = f_0 - \eta(1; kq, \overline{\chi}) + (\gamma + \sum_{p|kq} \frac{\log p}{p-1}) f_{-1},$$

$$\mathcal{E} \ll M^{\frac{1}{2}+\theta_F+\epsilon} q^{1+\theta_F+\epsilon} T \|x_n\|_1 \|y_n\|_\infty \left\| \frac{\tau_3 * |y|(n)}{n} \right\|_1 \left\| \frac{(\tau * |y|)(n)}{n} \right\|_1 \left\| \frac{y_n}{n} \right\|_1 \exp(-c\sqrt{\log T}),$$

$$\mathcal{E}' \ll q^{\theta_F+\epsilon} M^{\theta_F+\epsilon} T \left\| \frac{|x_s y_s|}{s} \right\|_1 \left\| \frac{(\tau_3 * |x|)(n)}{n} \right\|_1 \left\| \frac{y_v}{v} \right\|_1 \exp(-c\sqrt{\log T}).$$

If there exist some positive constant $a = a(\chi, F)$ such that both F_ψ and $L(s, \chi\psi)$ have no zeros in the region $\Re(s) \geq 1 - a$ for all ψ, then the term $\exp(-c\sqrt{\log T})$ in the error terms $\mathcal{E}, \mathcal{E}'$ can be replaced by $T^{-\delta+\epsilon}$ for some small enough $\delta = \delta(F, a) > 0$ uniformly for $M \leq \sqrt{T}$.

6.4.1. *Initial Manipulations.* The integral on the left (6.8) is

$$\begin{aligned} S_L &= \frac{1}{2\pi i}\int_{1-\kappa+iT_1}^{1-\kappa+iT_2} \frac{\overline{F}'}{\overline{F}}(1-s)B(s)L(1-s,\overline{\chi})X(s)Y(1-s)ds \\ &= \frac{1}{2\pi}\int_{T_1}^{T_2} \frac{\overline{F}'}{\overline{F}}(\kappa-it)B(1-\kappa+it)L(\kappa-it,\overline{\chi})X(1-\kappa+it)Y(\kappa-it)dt \\ &= \overline{\frac{1}{2\pi}\int_{T_1}^{T_2} \frac{F'}{F}(\kappa+it)\overline{B}(1-\kappa-it)L(\kappa+it,\chi)\overline{X}(1-\kappa-it)\overline{Y}(\kappa+it)dt} \\ &= \overline{\frac{1}{2\pi i}\int_{\kappa+iT_1}^{\kappa+iT_2} \frac{F'}{F}(s)\overline{B}(1-s)L(s,\chi)\overline{X}(1-s)\overline{Y}(s)ds} \\ &:= \overline{I_L}, \end{aligned}$$

where $\overline{B}(s) = \overline{B(\overline{s})}, \overline{X}(s) = \overline{X(\overline{s})}$ and $\overline{Y}(s) = \overline{Y(\overline{s})}$. Let

$$\frac{F'}{F}(s)L(s,\chi)\overline{Y}(s) = \sum_{m=1}^{\infty} a(m)m^{-s},$$

where

$$a(m) = -\sum_{uvw=m} \Lambda_F(u)\chi(v)\overline{y}_w. \tag{6.14}$$

Then,

$$I_L = \frac{1}{2\pi i}\sum_{k\leq M}\frac{\overline{x_k}}{k}\sum_{m=1}^{\infty} a(m)\int_{\kappa+iT_1}^{\kappa+iT_2} \overline{B}(1-s)\left(\frac{m}{k}\right)^{-s} ds.$$

Lemma 6.5. *Let x_n, y_n be supported on $n \leq M \leq T$. Then,*

$$I_L = \mathcal{M} + O\left(q^{1/2}T^{1/2}\mathcal{L}^3\|x_n\|_1\left\|\frac{\overline{y}_n}{n}\right\|_1 + q^{1/2+\theta_F+\epsilon}T^{1/2+\theta_F+\epsilon}M^{\theta_F+\epsilon}\|y_n\|_\infty\|x_n\|_1\right),$$

where

$$\mathcal{M} = \frac{\tau(\overline{\chi})}{q}\sum_{k\leq M}\frac{\overline{x_k}}{k}\sum_{m=\lceil\frac{kqT}{2\pi}\rceil}^{\frac{kqT}{\pi}} a(m)e\left(-\frac{m}{kq}\right). \tag{6.15}$$

To prove Lemma 6.5, we need the following lemmas.

Lemma 6.6. *For large A and $A < B \leq 2A$,*

$$\begin{aligned} &\int_A^B \exp\left(it\log\left(\frac{t}{re}\right)\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} dt \\ &= \begin{cases} (2\pi)^{1-a}r^a e^{-ir+\pi i/4} + E_a(r,A,B), & \text{if } A < r \leq B \leq 2A, \\ E_a(r,A,B), & \text{if } r \leq A \text{ or } r > B, \end{cases} \end{aligned}$$

where a is a fixed real number and

$$E_a(r,A,B) = O\left(A^{a-\frac{1}{2}}\right) + O\left(\frac{A^{a+\frac{1}{2}}}{|A-r|+A^{\frac{1}{2}}}\right) + O\left(\frac{B^{a+\frac{1}{2}}}{|B-r|+B^{\frac{1}{2}}}\right).$$

Proof. This is Lemma 2 in [26]. □

Lemma 6.7. *Let $r, \kappa_0 > 0, T_1 = T + O(1)$ and $T_2 = 2T + O(1)$. Then,*

$$\frac{1}{2\pi i}\int_{\kappa+iT_1}^{\kappa+iT_2} \overline{B}(1-s)r^{-s}ds = \frac{\tau(\overline{\chi})}{q}\delta_q(r)e_q(-r) + O(E(r,\kappa,T)r^{-\kappa}q^{\kappa-1/2}), \tag{6.16}$$

uniformly for $\kappa_0 \le \kappa \le 2$, where $\delta_q(r) = 1$ if $T_1/2\pi < r/q \le T_2/2\pi$ and 0 otherwise, and $E(r,\kappa,T) = E_\kappa\left(\frac{2\pi r}{q}, T_1, T_2\right) \ll T^{\kappa-\frac{1}{2}} + \frac{T^{\kappa+\frac{1}{2}}}{|T-\frac{2\pi r}{q}|+T^{1/2}} + \frac{T^{\kappa+\frac{1}{2}}}{|T-\frac{\pi r}{q}|+T^{1/2}}$.

Proof. From (6.4) and (6.10),

$$B(\sigma+it) = \left(\frac{q|t|}{2\pi}\right)^{\frac{1}{2}-\sigma-it}\exp\left(it + \operatorname{sign}(t)\frac{i\pi(1-2\mathfrak{a})}{4}\right)\left(\frac{\tau(\chi)}{i^{\mathfrak{a}}q^{\frac{1}{2}}} + O\left(\frac{1}{|t|}\right)\right).$$

Applying Lemma 6.6, for $T_1 < 2\pi r/q \le T_2$, we have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\kappa+iT_1}^{\kappa+iT_2} \overline{B}(1-s)r^{-s}ds \\
=&\frac{\exp(-\frac{i\pi(1-2\mathfrak{a})}{4})}{2\pi}\left(\frac{\overline{\tau(\chi)}}{i^{-\mathfrak{a}}q^{\frac{1}{2}}} + O\left(\frac{1}{T}\right)\right)\int_{T_1}^{T_2}\left(\frac{qt}{2\pi}\right)^{\kappa-\frac{1}{2}+it} r^{-\kappa-it}\exp(-it)dt \\
=&\frac{\exp(-\frac{i\pi(1-2\mathfrak{a})}{4})}{2\pi r^\kappa}\left(\frac{\overline{\tau(\chi)}}{i^{-\mathfrak{a}}q^{\frac{1}{2}}} + O\left(\frac{1}{T}\right)\right)\int_{T_1}^{T_2}\left(\frac{qt}{2\pi}\right)^{\kappa-\frac{1}{2}}\exp\left(it\log\left(\frac{qt}{2\pi re}\right)\right)dt \\
=&\frac{i^{\mathfrak{a}}}{2\pi r^\kappa}q^{\kappa-\frac{1}{2}}(2\pi)^{1-\kappa}\left(\frac{2\pi r}{q}\right)^\kappa\exp(-2\pi i r/q)\frac{\overline{\tau(\chi)}}{i^{-\mathfrak{a}}q^{\frac{1}{2}}} + E_\kappa\left(\frac{2\pi r}{q}, T_1, T_2\right)r^{-\kappa}q^{\kappa-1/2} \\
=&\frac{\chi(-1)\overline{\tau(\chi)}}{q}\exp\left(-2\pi i\frac{r}{q}\right) + E_\kappa\left(\frac{2\pi r}{q}, T_1, T_2\right)r^{-\kappa}q^{\kappa-1/2},
\end{aligned}$$

where

$$E_\kappa\left(\frac{2\pi r}{q}, T_1, T_2\right) \ll T^{\kappa-\frac{1}{2}} + \frac{T^{\kappa+\frac{1}{2}}}{|T-\frac{2\pi r}{q}|+T^{1/2}} + \frac{T^{\kappa+\frac{1}{2}}}{|T-\frac{\pi r}{q}|+T^{1/2}},$$

since $T_1 = T + O(1)$ and $T_2 = 2T + O(1)$. □

Proof. [Proof of Lemma 6.5]

Applying Lemma 6.7, we write

$$I_L = \mathcal{M} + \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3,$$

where

$$\mathcal{M} = \frac{\tau(\overline{\chi})}{q}\sum_{k\le M}\frac{\overline{x_k}}{k}\sum_{m=\lceil\frac{kqT}{2\pi}\rceil}^{\frac{kqT}{\pi}} a(m)e\left(-\frac{m}{kq}\right), \tag{6.17}$$

$$\mathcal{E}_0 \ll q^{-1/2}\sum_{k\le M}\frac{|\overline{x_k}|}{k}kq(kqT)^{\theta_F+\epsilon}\|y_k\|_\infty \ll q^{1/2+\theta_F+\epsilon}(MT)^{\theta_F+\epsilon}\|x_n\|_1\|y_n\|_\infty, \tag{6.18}$$

$$\mathcal{E}_1 \ll q^{1/2}T^{1/2}\sum_{k\le M}|\overline{x_k}|\sum_{m=1}^\infty\frac{|a(m)|}{m^\kappa}, \tag{6.19}$$

$$\mathcal{E}_2 \ll q^{1/2}T^{3/2}\sum_{k\le M}|\overline{x_k}|\sum_{m=1}^\infty\frac{|a(m)|}{m^\kappa}\left(\left|T-\frac{2\pi m}{qk}\right| + T^{1/2}\right)^{-1}, \tag{6.20}$$

$$\mathcal{E}_3 \ll q^{1/2}T^{3/2}\sum_{k\le M}|\overline{x_k}|\sum_{m=1}^\infty\frac{|a(m)|}{m^\kappa}\left(\left|T-\frac{\pi m}{qk}\right| + T^{1/2}\right)^{-1}. \tag{6.21}$$

For $\mathcal{E}_1$, we have

$$\mathcal{E}_1 \ll q^{1/2}T^{1/2}\|x_n\|_1 \sum_{m\le M}\frac{|\overline{y}_m|}{m^\kappa}\sum_{n=1}^{\infty}\frac{|\Lambda_F(n)|}{n^\kappa}\zeta(\kappa) \ll q^{1/2}T^{1/2}\mathcal{L}^3\|x_n\|_1\left\|\frac{\overline{y}_m}{m}\right\|_1. \tag{6.22}$$

The estimate for $\mathcal{E}_2$ and $\mathcal{E}_3$ is similar and we focus on $\mathcal{E}_2$. We write $\mathcal{E}_2 = \mathcal{E}_{21}+\mathcal{E}_{22}+\mathcal{E}_{23}$ corresponding to the following cases

(a) $\left|T-\frac{2\pi m}{qk}\right| > T/2$;

(b) $\sqrt{T} \le \left|T-\frac{2\pi m}{qk}\right| \le T/2$;

(c) $\left|T-\frac{2\pi m}{qk}\right| \le \sqrt{T}$.

In case (a), we have

$$\mathcal{E}_{21} \ll q^{1/2}\frac{T^{3/2}}{T}\sum_{k\le M}|\overline{x_k}|\sum_{m=1}^{\infty}\frac{|a(m)|}{m^\kappa} \ll \mathcal{E}_1 \ll q^{1/2}T^{1/2}\mathcal{L}^3\|x_n\|_1\left\|\frac{\overline{y}_m}{m}\right\|_1.$$

In case (b), without loss of generality we can assume $\sqrt{T} \le \frac{2\pi m}{qk} - T \le T/2$. We can divide this range into $\ll \log T$ intervals of the form $T+P < 2\pi m/kq \le T+2P$ with $\sqrt{T} \ll P \ll T$. Thus, m lies in intervals of the form $I := [\frac{qk}{2\pi}(T+P), \frac{qk}{2\pi}(T+2P)]$. From (6.14), we have $|a(m)| = |\sum_{uvw=m}\Lambda_F(u)\chi(v)\overline{y}_w| \ll d_F\tau_3(m)m^{\theta_F}\log m\|y_n\|_\infty$, and this gives

$$\begin{aligned}
\mathcal{E}_{22} &\ll q^{1/2}T^{3/2}\sum_{k\le M}|\overline{x_k}|\sum_P\sum_{m\in I}\frac{|a(m)|}{m^\kappa P}\\
&\ll q^{1/2}T^{3/2}\sum_P\sum_{k\le M}\frac{|\overline{x_k}|}{qk}\sum_{m\in I}\frac{|a(m)|}{TP}\\
&\ll q^{1/2}T^{1/2}\|y_n\|_\infty\sum_P\sum_{k\le M}\frac{|\overline{x_k}|}{qkP}\sum_{m\in I}\tau_3(m)m^{\theta_F}\log m\\
&\ll q^{1/2}T^{1/2}\|y_n\|_\infty\sum_P\sum_{k\le M}\frac{|\overline{x_k}|}{qkP}qkP(qMT)^{\theta_F+\epsilon}\\
&\ll q^{1/2+\theta_F+\epsilon}T^{1/2+\theta_F+\epsilon}M^{\theta_F+\epsilon}\|y_n\|_\infty\|x_n\|_1.
\end{aligned}$$

For case (c), we have $|T-\frac{2\pi m}{qk}| \le \sqrt{T}$, thus m lies in intervals of the form

$$J := [\frac{qk}{2\pi}(T-\sqrt{T}), \frac{qk}{2\pi}(T+\sqrt{T})],$$

which gives

$$\begin{aligned}
\mathcal{E}_{23} &\ll q^{1/2}T^{3/2}\sum_{k\le M}|\overline{x_k}|\sum_{m\in J}\frac{|a(m)|}{qkT}\frac{1}{\sqrt{T}}\\
&\ll q^{1/2}\|y_n\|_\infty\sum_{k\le M}|\overline{x_k}|\sum_{m\in J}\frac{\tau_3(m)m^{\theta_F}\log m}{qk}\\
&\ll q^{1/2}\|y_n\|_\infty\|x_n\|_1\frac{1}{qk}qk\sqrt{T}(qMT)^{\theta_F+\epsilon}\\
&\ll q^{1/2+\theta_F+\epsilon}T^{1/2+\theta_F+\epsilon}M^{\theta_F+\epsilon}\|y_n\|_\infty\|x_n\|_1.
\end{aligned}$$

Combining all three cases, we have

$$\mathcal{E}_2 \ll q^{1/2}T^{1/2}\mathcal{L}^3\|x_n\|_1\left\|\frac{\overline{y}_m}{m}\right\|_1 + q^{1/2+\theta_F+\epsilon}T^{1/2+\theta_F+\epsilon}M^{\theta_F+\epsilon}\|y_n\|_\infty\|x_n\|_1. \tag{6.23}$$

Lemma 6.5 thus follows from (6.17), (6.22), and (6.23). □

6.4.2. *Main Term Set up.* We want to evaluate the main term $\mathcal{M}$, which is given in (6.15) by

$$\mathcal{M} = \frac{\tau(\overline{\chi})}{q} \sum_{k \leq M} \frac{\overline{x_k}}{k} \sum_{m=\lceil \frac{kqT}{2\pi} \rceil}^{\frac{kqT}{\pi}} a(m) e\left(-\frac{m}{kq}\right). \tag{6.24}$$

Write $\frac{m}{qk} = \frac{m'}{k'}$, where $(m', k') = 1$. Then

$$e\left(-\frac{m}{qk}\right) = \frac{1}{\phi(k')} \sum_{\psi \pmod{k'}} \tau(\overline{\psi}) \psi(-m'). \tag{6.25}$$

Next, we write the sum over ψ in terms of primitive characters. If $\psi \mod k'$ is induced by the primitive character $\tilde{\psi} \mod g$, then (cf. [18, p. 67])

$$\tau(\psi) = \mu\left(\frac{k'}{g}\right) \tilde{\psi}\left(\frac{k'}{g}\right) \tau(\tilde{\psi}). \tag{6.26}$$

Thus,

$$\begin{aligned}
e\left(-\frac{m}{qk}\right) &= \frac{1}{\phi(k')} \sum_{\psi \mod k'} \tau(\overline{\psi}) \psi(-m') \\
&= \frac{1}{\phi(k')} \sum_{g | k'} \sideset{}{^*}\sum_{\tilde{\psi} \mod g} \mu\left(\frac{k'}{g}\right) \overline{\tilde{\psi}}\left(\frac{k'}{g}\right) \tau(\overline{\tilde{\psi}}) \tilde{\psi}(-m') \qquad (6.27) \\
&= \frac{\tau(\chi)\mu(\frac{k'}{q})\chi(\frac{k'}{q})\overline{\chi}(-m')}{\phi(k')} 1_{q|k'} \qquad (6.28) \\
&+ \frac{\tau(\overline{\psi^*})\mu(\frac{k'}{g^*})\overline{\psi^*}(\frac{k'}{g^*})\psi^*(-m')}{\phi(k')} 1_{g^*|k'} \qquad (6.29) \\
&+ \frac{1}{\phi(k')} \sum_{g | k'} \sideset{}{^*}\sum_{\substack{\tilde{\psi} \pmod{g} \\ \tilde{\psi} \neq \overline{\chi}, \psi^*}} \mu\left(\frac{k'}{g}\right) \overline{\tilde{\psi}}\left(\frac{k'}{g}\right) \tau(\overline{\tilde{\psi}}) \tilde{\psi}(-m'), \qquad (6.30)
\end{aligned}$$

where $\sum^*$ denotes the sum is over primitive characters. Using the Möbius inversion formula for an arbitrary function f (cf. [17, eq. (5.10)]), we have

$$\begin{aligned}
f(m', k') &= f\left(\frac{m}{(m, kq)}, \frac{kq}{(m, kq)}\right) \\
&= \sum_{d | (m, kq)} \sum_{e | d} \mu\left(\frac{d}{e}\right) f\left(\frac{m}{e}, \frac{kq}{e}\right),
\end{aligned}$$

and we obtain

$$
\begin{aligned}
&e\left(-\frac{m}{qk}\right) - \frac{\tau(\chi)\mu(\frac{k'}{q})\chi(\frac{k'}{q})\overline{\chi}(-m')}{\phi(k')}1_{q|k'} - \frac{\tau(\overline{\psi^*})\mu(\frac{k'}{g})\overline{\psi^*}(\frac{k'}{g})\psi^*(-m')}{\phi(k')}1_{g^*|k'}\\
&= \sum_{\substack{d|m\\ d|kq}} \sum_{e|d} \frac{\mu(d/e)}{\phi(kq/e)} \sum_{g|kq/e} \sideset{}{^*}\sum_{\substack{\psi \bmod g\\ \psi\neq\overline{\chi}\\ \psi\neq\psi^*}} \mu\left(\frac{kq}{eg}\right)\overline{\psi}\left(\frac{kq}{eg}\right)\tau(\overline{\psi})\psi\left(-\frac{m}{e}\right)\\
&= \sum_{g|kq} \sideset{}{^*}\sum_{\substack{\psi \pmod g\\ \psi\neq\overline{\chi}\\ \psi\neq\psi^*}} \tau(\overline{\psi}) \sum_{\substack{d|m\\ d|kq}} \sum_{\substack{e|d\\ e|kq/g}} \frac{\mu(d/e)}{\phi(kq/e)}\overline{\psi}\left(-\frac{kq}{eg}\right)\psi\left(\frac{m}{e}\right)\mu\left(\frac{kq}{eg}\right)\\
&= \sum_{g|kq} \sideset{}{^*}\sum_{\substack{\psi \pmod g\\ \psi\neq\overline{\chi}\\ \psi\neq\psi^*}} \tau(\overline{\psi}) \sum_{\substack{d|m\\ d|kq}} \psi\left(\frac{m}{d}\right)\delta(g,kq,d,\psi),
\end{aligned}
$$

where

$$
\delta(g,kq,d,\psi) = \sum_{\substack{e|d\\ e|kq/g}} \frac{\mu(d/e)}{\phi(kq/e)}\overline{\psi}\left(-\frac{kq}{eg}\right)\psi\left(\frac{d}{e}\right)\mu\left(\frac{kq}{eg}\right). \tag{6.31}
$$

We write $\mathcal{M}$ as

$$
\mathcal{M} = \mathcal{M}_{\overline{\chi}} + \mathcal{M}_{\psi^*} + \mathcal{E} \tag{6.32}
$$

where

$$
\mathcal{M}_{\overline{\chi}} = \frac{\tau(\overline{\chi})}{q}\sum_{k\le M}\frac{\overline{x_k}}{k}\sum_{kqT/2\pi\le m\le kqT/\pi} a(m)\frac{\tau(\chi)\mu(\frac{k'}{q})\chi(\frac{k'}{q}))\overline{\chi}(-m')}{\phi(k')}1_{q|k'} \tag{6.33}
$$

$$
\mathcal{M}_{\psi^*} = \frac{\tau(\overline{\chi})}{q}\sum_{k\le M}\frac{\overline{x_k}}{k}\sum_{kqT/2\pi\le m\le kqT/\pi} a(m)\frac{\tau(\overline{\psi^*})\mu(\frac{k'}{g^*})\overline{\psi^*}(\frac{k'}{g^*})\psi^*(-m')}{\phi(k')}1_{g^*|k'} \tag{6.34}
$$

$$
\mathcal{E} = \frac{\tau(\overline{\chi})}{q}\sum_{k\le M}\frac{\overline{x_k}}{k}\sum_{kqT/2\pi\le m\le kqT/\pi} a(m)\sum_{\ell|q}\sum_{g|k}\sideset{}{^*}\sum_{\substack{\psi \pmod{g\ell}\\ \psi\neq\overline{\chi}\\ \psi\neq\psi^*}}\tau(\overline{\psi})\sum_{\substack{d|m\\ d|kq}}\psi\left(\frac{m}{d}\right)\delta(g\ell,kq,d,\psi). \tag{6.35}
$$

Here we have used the fact that $(k,q)=1$ to rewrite $\psi \pmod g$ with $g \mid kq$ as $\psi \pmod{g\ell}$ with $\ell \mid q$ and $g \mid k$ in $\mathcal{E}$. To evaluate (6.33), (6.34) and (6.35), we need the following lemma.

Lemma 6.8. *Let $F \in \mathcal{S}^*$, χ be a non-real primitive character modulo $q > 1$ and ψ a primitive character modulo g. If g is squarefree, $g \le Q$ and $(g, B_Q) = 1$, then for positive integers h, k and $Q \le \exp(2\sqrt{\log x})$, we have*

$$
\sum_{\substack{u\le x\\ (u,k)=1}} \Lambda_F * \chi(hu)\psi(u) = R(x,h,k,\psi) + O\left(h^{\theta_F}\tau(h)j(h)j(k)\log k(\log h)^3 x\exp(-c\sqrt{\log x})\right),
$$

where

$$
\begin{aligned}
&R(x,h,k,\psi)\\
&= \begin{cases} \frac{\phi(k)(k,q)}{k\phi((k,q))}\frac{\phi(q)}{q}x\left(f_{-1}\chi(h)\log(x/e) + X_1(h,k) + \chi(h)X_2(k)\right), & \text{if } \psi = \overline{\chi},\\ \chi(h)x\Phi(1;k,\chi\psi)L(1,\chi\psi)f_{-1}, & \text{if } \psi \neq \overline{\chi},\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&G(z,\psi) = \sum_{d=1}^{\infty} \frac{\Lambda_F(d)\psi(d)}{d^z},\\
&f_{-1} = \mathrm{Res}_{z=1}\, G(z), \qquad f_0 = \lim_{z\to 1} G(z) - \frac{f_{-1}}{z-1},\\
&\eta(z;p,l,\psi) = \sum_{m=1}^{\infty} \frac{\Lambda_F(p^{l+m})\psi(p^m)}{p^{mz}}, \qquad \eta(z;k,\psi) = \sum_{p|k} \eta(z,p,0,\psi),\\
&X_1(h,k) = \sum_{a|(h,k)} \Lambda_F(a)\chi(h/a) + \sum_{\substack{p^l|h, p|q\\(p,k)=1}} \chi(h/p^l)\Lambda_F(p^l)\\
&\qquad + \chi(h) \sum_{\substack{p^l|h\\(p,kq)=1}} \overline{\chi}(p^l)\eta(1;p,l,\psi)(1-p^{-1}),\\
&X_2(k) = f_0 - \eta(1;k,\psi) + \Big(\gamma + \sum_{p|kq} \frac{\log p}{p-1}\Big) f_{-1},\\
&j(n) = \prod_{p|n} (1+10p^{-1/2}). \qquad (6.36)
\end{aligned}
$$

Here c is some positive absolute constant depending only on F and χ. If there exists some absolute constant $a>0$ such that $L(s,\chi\psi)$ and F_ψ have no zeros in the region $\Re(s) \geq 1-a$ for all ψ, then the error term can be replaced by

$$
O\left(h^{\theta_F}\tau(h)j(h)j(k)\log k(\log h)^3 x^{1-\delta+\epsilon}\right),
$$

for some small enough $\delta = \delta(F,a) > 0$ uniformly for $Q \leq x$.

To prove Lemma 6.8, we need the following lemmas.

Lemma 6.9 (Decomposition of convolutions, [38, Lemma 6.1])**.** *Let $j, D \in \mathbb{N}$ and let $f_1, \ldots, f_j$ be arithmetic functions. Given a decomposition of integers $D = \prod_{i=1}^{j} d_i$, define the integers $D_i = \prod_{u=1}^{j-i} d_u$ for $1 \leq i \leq j-1$ and $D_j = 1$. Then we have the following identities:*

$$
\sum_{\substack{m\leq X\\(m,k)=1}} (f_1 * \cdots * f_j)(mD) = \sum_{d_1\cdots d_j = D} \sum_{\substack{m_1\cdots m_j \leq X\\(m_i,kD_i)=1}} f_1(m_1 d_j) f_2(m_2 d_{j-1}) \cdots f_j(m_j d_1), \qquad (6.37)
$$

$$
\sum_{(m,k)=1} \frac{(f_1 * \cdots * f_j)(mD)}{m^s} = \sum_{d_1 d_2\cdots d_j = D} \prod_{i=1}^{j} \sum_{(m_i,kD_i)=1} \frac{f_i(m_i d_{j-i+1})}{m_i^s}. \qquad (6.38)
$$

Lemma 6.10 (A variant of Perron's Formula, [35, Theorem 2.1])**.** *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with abscissa of absolute convergence σ_a. Let*

$$
B(\sigma) = \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma}, \qquad (6.39)
$$

for $\sigma > \sigma_a$. Then for $b > \sigma_a$, $x \geq 2$, $U \geq 2$, and $H \geq 2$, we have

$$
\sum_{n\leq x} a_n = \frac{1}{2\pi i}\int_{b-iU}^{b+iU} f(s)\frac{x^s}{s}ds + O\left(\sum_{x-x/H\leq n\leq x+x/H} |a_n|\right) + O\left(\frac{x^b H B(b)}{U}\right). \qquad (6.40)
$$

Lemma 6.11 (Dirichlet's Hyperbola Principle in short intervals, [12, Lemma 4.1])**.** *Let $x, y, z \in \mathbb{R}$ such that $1 \le \max(y, \frac{x}{y}) \le z \le x$. Then for any arithmetic function f and g*

$$\sum_{x \le n \le x+y} (f * g)(n) = \sum_{d \le z} f(d) \sum_{\frac{x}{d} \le k \le \frac{x+y}{d}} g(k) + \sum_{k \le x/z} \sum_{\frac{x}{k} \le d \le \frac{x+y}{k}} g(d) + O\left(\max_{k \le 2x/z} |g(k)| \sum_{z \le d \le z(1+y/x)} |f(d)| \right).$$

Now we are ready to prove Lemma 6.8.

Proof. [Proof of Lemma 6.8] Let

$$A(z; h, \psi) = A(z) = \sum_{\substack{u=1 \\ (u,k)=1}}^{\infty} \Lambda_F * \chi(hu)\psi(u)u^{-z}.$$

From Lemma 3.2, we have for $\kappa = 1 + O((\log x)^{-1})$ and $h \le x$

$$\sum_{n=1}^{\infty} \frac{|\Lambda_F * \chi(hu)\psi(u)|}{u^\kappa} \le \sum_u \frac{(|\Lambda_F| * 1)(hu)}{u^\kappa} = \sum_{h_1 h_2 = h} \sum_{(u_1, h_1)=1} \frac{1}{u_1^\kappa} \sum_{u_2} \frac{|\Lambda_F(u_2 h_1)|}{u_2^\kappa}$$
$$\ll \tau(h) h^\kappa \sum_n \frac{(|\Lambda_F| * 1)(n)}{n^\kappa} \ll \tau(h) h (\log x)^2.$$

Therefore, taking $b = \kappa$ and $H = \sqrt{U}$ in Lemma 6.10, we have

$$\sum_{\substack{u \le x \\ (u,k)=1}} \Lambda_F * \chi(hu)\psi(u)$$
$$= \frac{1}{2\pi i} \int_{\kappa - iU}^{\kappa + iU} A(z) x^z \frac{dz}{z} + O\left(\sum_{x - x/\sqrt{U} \le u \le x + x/\sqrt{U}} |\Lambda_F * \chi(hu)\psi(u)| \right) + O\left(\frac{\tau(h) h x (\log x)^2}{\sqrt{U}} \right).$$

Note that

$$\sum_{x - x/\sqrt{U} \le u \le x + x/\sqrt{U}} |\Lambda_F * \chi(hu)\psi(u)|$$
$$\ll \sum_{x - x/\sqrt{U} \le n \le x + x/\sqrt{U}} \sum_{d | hn} |\Lambda_F(d)|$$
$$\ll \sum_{x - x/\sqrt{U} \le n \le x + x/\sqrt{U}} \left(\sum_{d|h} |\Lambda_F(d)| + \sum_{d|n} |\Lambda_F(d)| \right)$$
$$\ll \frac{x}{\sqrt{U}} \tau(h) h^{\theta_F} \log h + \sum_{x - x/\sqrt{U} \le n \le x + x/\sqrt{U}} |(1 * \Lambda_F)(n)|.$$

Using property (iv) of $\mathcal{S}^*$ and partial summation,

$$\sum_{n \le x} \frac{|\Lambda_F(n)|}{n} \le \left(\sum_{n \le x} \frac{\Lambda(n)}{n} \right)^{1/2} \left(\sum_{n \le x} \frac{\Lambda(n) |\lambda_F(n)|^2}{n} \right)^{1/2} \ll \log x. \tag{6.41}$$

We also have, from property (v) of $\mathcal{S}^*$, that for $x \gg_F U^c$

$$\sum_{x \le n \le x + \frac{x}{\sqrt{U}}} |\Lambda_F(n)| \ll \sum_{x \le n \le x e^{1/\sqrt{U}}} |\Lambda_F(n)| \ll \frac{x}{\sqrt{U}}. \tag{6.42}$$

We apply Lemma 6.11 with $f = |\Lambda_F|, g = 1$, $y = z = 2x/\sqrt{U}$. Then for $\eta = \eta(F) > 0$ small enough, we have uniformly for $U \le x^\eta$,

$$x \gg U^c, z \gg \left(\frac{x}{y}\right)^c, \tag{6.43}$$

and thus

$$\begin{aligned}
&\sum_{x - x/\sqrt{U} \le n \le x + x/\sqrt{U}} |(\Lambda_F * 1)(n)| \\
&= \sum_{d \le z} |\Lambda_F(d)| \sum_{\frac{x-x/\sqrt{U}}{d} \le k \le \frac{x+x/\sqrt{U}}{d}} 1 + \sum_{k \le x/z} \sum_{\frac{x-x/\sqrt{U}}{k} \le d \le \frac{x+x/\sqrt{U}}{k}} |\Lambda_F(d)| + O\left(\sum_{z \le d \le z(1+\frac{y}{x - \frac{x}{\sqrt{U}}})} |\Lambda_F(d)| \right) \\
&\ll y \sum_{d \le z} \frac{|\Lambda_F(d)|}{d} + \sum_{k \le x/z} \frac{y}{k} + O\left(\frac{yz}{x - x/\sqrt{U}}\right) \\
&\ll y \log z + y \log x + z \\
&\ll \frac{x}{\sqrt{U}} \log x.
\end{aligned}$$

Therefore, for $h \le x$ and $U \le x^\eta$,

$$\sum_{\substack{u \le x \\ (u,k)=1}} \Lambda_F * \chi(hu)\psi(u) = \frac{1}{2\pi i} \int_{\kappa - iU}^{\kappa + iU} A(z) x^z \frac{dz}{z} + O\left(\tau(h) h \frac{x(\log x)^2}{\sqrt{U}}\right). \tag{6.44}$$

Applying Lemma 6.9, we write

$$A(z) = \sum_{ab=h} \sum_{(c,ak)=1} \frac{\chi(bc)\psi(c)}{c^z} \sum_{(d,k)=1} \frac{\Lambda_F(ad)\psi(d)}{d^z} := \sum_{ab=h} A_1(z;a,b) A_2(z;a) \tag{6.45}$$

where

$$A_1(z;a,b) = \chi(b) \sum_{(c,ak)=1} \frac{\chi\psi(c)}{c^z} = \chi(b) L(s, \chi\psi) \prod_{p|ak} (1 - \chi\psi(p) p^{-z}),$$

and

$$A_2(z;a) = \begin{cases} \sum_{(d,k)=1} \frac{\Lambda_F(d)\psi(d)}{d^z}, & \text{if } a = 1, \\ \sum_{k=0}^{\infty} \frac{\Lambda_F(p^{l+k})\psi(p^k)}{p^{kz}}, & \text{if } a = p^l, p \nmid k, \\ \Lambda_F(a), & \text{if } a = p^l, p \mid k, \\ 0, & \text{else.} \end{cases}$$

Using the following notation

$$\eta(z;p,l,\psi) = \sum_{k=0}^{\infty} \frac{\Lambda_F(p^{l+k})\psi(p^k)}{p^{kz}}, \tag{6.46}$$

$$\Phi(z;k,\chi) = \prod_{p|k} (1 - \chi(p) p^{-z}), \tag{6.47}$$

$$G(z;\psi) = \sum_{d=1}^{\infty} \frac{\Lambda_F(d)\psi(d)}{d^z} = \sum_p \eta(z;p,0,\psi), \tag{6.48}$$

$$\eta(z;k,\psi) = \sum_{p|k} \sum_{m=1}^{\infty} \frac{\Lambda_F(p^m)\psi(p^m)}{p^{mz}} = \sum_{p|k} \eta(z;p,0,\psi), \tag{6.49}$$

we can represent the $A(z)$ in (6.45) as

$$\begin{aligned}A(z) &= \chi(h)L(s,\chi\psi)\Phi(z;k,\chi\psi)\left(G(z;\psi)-\eta(z;k,\psi)\right)\\ &\quad+\sum_{p^l|h,p\nmid k}\chi(h/p^l)L(s,\chi\psi)\Phi(z;p^lk,\chi\psi)\eta(z;p,l,\psi)\\ &\quad+\sum_{p^l|(h,k)}\chi(h/p^l)L(s,\chi\psi)\Phi(z;p^lk,\chi\psi)\Lambda_F(p^l)\\ &:= B_1(z)+B_2(z)+B_3(z).\end{aligned}$$

From $F\in\mathcal{S}^*$ and Lemma 3.2, we deduce that $G(z;\psi)$ has at most a simple pole at $z=1$. This shows that $A(z)$ has a pole at $z=1$ of order at most 2. Hence,

$$\frac{1}{2\pi i}\int_{\kappa-iU}^{\kappa+iU}A(z)x^z\frac{dz}{z}=\mathrm{Res}_{z=1}\,A(z)x^zz^{-1}+\frac{1}{2\pi i}\int_{\sigma_0(U)-iU}^{\sigma_0(U)+iU}A(z)x^z\frac{dz}{z} \tag{6.50}$$

$$+\left(\int_{\sigma_0(U)+iU}^{\kappa+iU}+\int_{\kappa-iU}^{\sigma_0(U)-iU}\right)A(z)x^z\frac{dz}{z}, \tag{6.51}$$

where $\sigma_0(U)=1-\frac{c}{\log(Q(U+2))}$ for some positive constant $c=c(F)$. From Lemma 3.2 and [18, Ch 14, eq (13)], we have

$$\begin{aligned}|G(z,\psi)|&\ll\log^2(Q(|z|+2)),\quad\text{for}\quad\Re(z)\geq 1-c/\log(Q(|\Im(z)|+2)),\\ |L(z,\chi\psi)|&\ll\log(Q(|z|+2)),\quad\text{for}\quad\Re(z)\geq 1-c/\log(Q(|\Im(z)|+2)).\end{aligned}$$

From (6.36) we have $j(n)\ll\exp\left(o\left(\sqrt{\log n}\right)\right)$ and $\Phi(z;k,\chi)\ll j(k)$, for $\Re(z)\geq 1-c/\log(Q(U+2))$. Moreover, since $|\Lambda_F(p^j)|\ll_F jp^{j\theta_F}\log p$, we see that

$$\eta(z;p,l,\psi)=\sum_{r=0}^{\infty}\frac{\Lambda_F(p^{l+r})\psi(p^r)}{p^{rz}}\ll\sum_{r=0}^{\infty}\frac{(r+l)p^{(r+l)\theta_F}\log p}{p^{rz}}\ll(l+1)p^{l\theta_F}\log p, \tag{6.52}$$

and it follows that

$$\begin{aligned}\eta(z;k,\psi)&\ll\sum_{p|k}\eta(z;p,0,\psi)\ll\sum_{p|k}\log p\ll\log k,\\ B_2(z)&\ll\sum_{p^l|h}j(hk)(l+1)p^{l\theta_F}\log p\log(Q(|\Im z|+2))\ll j(hk)h^{\theta_F}(\log h)^3\log(Q(|\Im z|+2)),\\ B_3(z)&\ll\sum_{p^l|(h,k)}j(hk)\log(Q(|\Im z|+2))h^{\theta_F}\log h\ll j(hk)\tau(h)h^{\theta_F}\log h\log(Q(|\Im z|+2)).\end{aligned}$$

This shows that for $\Re(z)\geq\sigma_0(U)$ and $|\Im z|\leq U$,

$$\begin{aligned}A(z)&=B_1(z)+B_2(z)+B_3(z)\\ &\ll j(k)\log(QU)\left(\log^2(QU)+\log k\right)+h^{\theta_F}j(hk)\log(QU)(\log h)^3+\tau(h)h^{\theta_F}j(hk)\log h\log(QU)\\ &\ll\tau(h)h^{\theta_F}j(h)j(k)\log k(\log h)^3(\log(QU))^3.\end{aligned}$$

Thus, the horizontal integrals in (6.51) are bounded by

$$\int_{\sigma_0(U)}^{\kappa}\frac{A(\sigma\pm iU)}{|\sigma\pm iU|}x^\sigma d\sigma\ll\frac{x\tau(h)j(h)h^{\theta_F}j(k)\log k(\log h)^3(\log(QU))^3}{U}.$$

The left vertical integral in (6.50) is bounded by

$$\begin{aligned}x^{\sigma_0(U)}\int_{-U}^{U}\frac{A(\sigma+iu)}{|\sigma+iu|}du&\ll x^{\sigma_0(U)}h^{\theta_F}\tau(h)j(h)j(k)\log k(\log h)^3(\log(QU))^4\\ &\ll xh^{\theta_F}\tau(h)j(h)j(k)\log k(\log h)^3(\log(QU))^4\exp\left(-\frac{c\log x}{\log(Q(|U|+2))}\right).\end{aligned}$$

If we choose $U=\exp(c'\sqrt{\log x})$, then we have uniformly for $Q\le \exp(2\sqrt{\log x})$ all integrals in (6.50) and (6.51) bounded by

$$O\left(h^{\theta_F}\tau(h)j(h)j(k)\log k(\log h)^3 x\exp(-c''\sqrt{\log x})\right).$$

If $L(s,\chi\psi)$ and F_ψ have no zeros in the region $\Re(s)\ge 1-a$, then we can choose $\sigma_0(U)=1-a+\epsilon$. We have by Lemma 3.2 $\sigma_0(U)\le\sigma\le\kappa$,

$$G(z,\psi)\ll(\log QU)^2. \tag{6.53}$$

We also have for $\sigma\ge\sigma_0(U)$, $L(\sigma+it,\chi\psi)\ll(QU)^\epsilon$, if $L(s,\chi\psi)$ has no zeros for $\Re(s)\ge 1-a$. Thus,

$$A(z)\ll\tau(h)h^{\theta_F}j(k)\log k(\log h)^3(\log(QU))^3(QU)^\epsilon. \tag{6.54}$$

Therefore, we have

$$\int_{\sigma_0(U)}^{\kappa}\frac{A(\sigma\pm iU)}{|\sigma\pm iU|}x^\sigma d\sigma\ll\frac{x\tau(h)j(h)h^{\theta_F}j(k)\log k(\log h)^3(\log QU)^3(QU)^\epsilon}{U}, \tag{6.55}$$

$$x^{\sigma_0(U)}\int_{-U}^{U}\frac{A(\sigma+iu)}{|\sigma+iu|}du\ll x^{1-a}h^{\theta_F}\tau(h)j(h)j(k)\log k(\log h)^3(\log QU)^4(QU)^\epsilon. \tag{6.56}$$

Combining these with (6.44), we see that taking $U=x^{\min(2a,\eta)}$ yields an error term of size

$$O\left(h^{\theta_F}\tau(h)j(h)j(k)\log k(\log h)^3x^{1-\delta+\epsilon}\right),$$

uniformly for $Q\le x$. Next we compute the residue at $z=1$. Suppose we have the Laurent series

$$\begin{aligned}
&G(z)=\frac{f_{-1}}{(z-1)}+f_0+f_1(z-1)+f_2(z-1)^2+\cdots,\\
&L(z,\chi\psi)=\frac{c_{-1}}{(z-1)}+c_0+c_1(z-1)+c_2(z-1)^2+\cdots,\\
&\Phi(z;k,\chi\psi)=\Phi(1;k,\chi\psi)+\Phi'(1;k,\chi\psi)(z-1)+\frac{1}{2}\Phi^{(2)}(1;k,\chi\psi)(z-1)^2+\cdots,\\
&\eta(z;k,\psi)=\eta(1;k,\psi)+\eta'(1;k,\psi)(z-1)+\eta^{(2)}(1;k,\psi)(z-1)^2+\cdots,\\
&\frac{x^z}{z}=x\left(1+\log(x/e)(z-1)+\left(\frac{1}{2}\log^2 x-\log(x/e)\right)(z-1)^2+\cdots\right).
\end{aligned}$$

From the fact that

$$\Phi(1;p^lk,\chi\psi)=\begin{cases}\Phi(1;k,\chi\psi), & p\mid k,\\ \Phi(1;k,\chi\psi)(1-\chi\psi(p)p^{-1}), & p\nmid k.\end{cases}$$

we see that,

$$\begin{aligned}
&\operatorname{Res}_{z=1}A(z)x^zz^{-1}\\
&=\chi(h)x\Phi(1;k,\chi\psi)\left(c_{-1}(f_0-\eta(1;k,\psi))+c_0f_{-1}\right)\\
&\quad+\chi(h)x(\log(x/e)\Phi(1;k,\chi\psi)+\Phi'(1;k,\chi\psi))f_{-1}c_{-1}\\
&\quad+x\Phi(1;k,\chi\psi)c_{-1}\left(\sum_{a\mid(h,k)}\Lambda_F(a)\chi(h/a)+\sum_{p^l\mid h,p\nmid k}\chi(h/p^l)\eta(1;p,l,\psi)(1-\chi\psi(p)p^{-1})\right).
\end{aligned}$$

If $\chi\psi$ is principal, then $L(s,\chi\psi)=\prod_{p|q}(1-p^{-s})\zeta(s)$ and thus $c_{-1}=\frac{\phi(q)}{q}$ and $c_0=\frac{\phi(q)}{q}(\gamma+\sum_{p|q}\frac{\log p}{p-1})$. From (6.47), we have

$$\Phi(1;k,\chi\psi)=\prod_{p|k,p\nmid q}(1-p^{-1})=\frac{\phi(k)(k,q)}{k\phi((k,q))},$$
$$\Phi'(1;k,\chi\psi)=\Phi(1;k,\chi\psi)\sum_{p|k,p\nmid q}\frac{\log p}{p-1}.$$

Since $\psi(p)=0$ for $p\mid q$, we see that for $p\mid q$,

$$\eta(z;p,l,\psi)=\Lambda_F(p^l), \tag{6.57}$$

and thus the residue of $A(z)$ at $z=1$ can be written as

$$\begin{aligned}&\mathrm{Res}_{z=1}A(z)x^zz^{-1}\\ =&x\chi(h)\frac{\phi(k)(k,q)}{k\phi((k,q))}\frac{\phi(q)}{q}\left(f_0-\eta(1;k,\psi)+\Big(\gamma+\sum_{p|q}\frac{\log p}{p-1}\Big)f_{-1}\right)\\ &+x\frac{\phi(k)(k,q)}{k\phi((k,q))}\frac{\phi(q)}{q}\left(\chi(h)\Big(\log(x/e)+\sum_{\substack{p|k\\ p\nmid q}}\frac{\log p}{p-1}\Big)f_{-1}+\sum_{a|(h,k)}\Lambda_F(a)\chi(h/a)\right)\\ &+x\frac{\phi(k)(k,q)}{k\phi((k,q))}\frac{\phi(q)}{q}\left(\sum_{\substack{p^l|h,p|q\\ (p,k)=1}}\chi(h/p^l)\Lambda_F(p^l)+\sum_{\substack{p^l|h\\ (p,kq)=1}}\chi(h/p^l)\eta(1;p,l,\psi)(1-p^{-1})\right).\end{aligned}$$

If $\chi\psi$ is non-principal, the only possible pole of $A(z)$ arises from $G(z;\psi)$, in which case it is a simple pole at $z=1$, and

$$\mathrm{Res}_{z=1}A(z)x^zz^{-1}=x\chi(h)\Phi(1;k,\chi\psi)L(1,\chi\psi)f_{-1}.$$

□

6.5. **Proof of Theorem 6.4.** To evaluate (6.33), (6.34) and (6.35), we apply Lemma 6.8 with $x=\frac{qkT}{2\pi dv}$. First we note that $\frac{kqT}{2\pi dv}\gg T^{1/2}$ since $d\le M\le\sqrt{T}$. By the support of x_n, we have $G(z,\psi)$ have zero free region of the form $1-\sigma\ll\frac{1}{\log M(|\Im z|+2)}$ for all ψ with conductor $\le M$. Therefore, we all terms with $\exp(-c\sqrt{\log x})$ (or $x^{-\delta+\epsilon}$) can be adjusted to some absolute constant c (or δ) depending on F and $L(s,\chi)$ uniformly for $M\le\exp(\sqrt{\log x})$ (or $M\le\sqrt{T}$).

6.5.1. *Error Terms.* We change the order of summation and expand the definition of $a(m)$ (6.14) to obtain

$$\begin{aligned}\mathcal{E}=&\frac{\tau(\overline{\chi})}{q}\sum_{k\le M}\frac{\overline{x_k}}{k}\sum_{\ell|q}\sum_{g|k}\ \sideset{}{^*}\sum_{\substack{\psi\ (\mathrm{mod}\ g\ell)\\ \psi\ne\overline{\chi}\\ \psi\ne\psi^*}}\tau(\overline{\psi})\sum_{d|kq}\delta(g\ell,kq,d,\psi)\sum_{kqT/2\pi d\le m\le kqT/\pi d}a(md)\psi(m)\\ =&-\frac{\tau(\overline{\chi})}{q}\sum_{k\le M}\frac{\overline{x_k}}{k}\sum_{\ell|q}\sum_{g|k}\ \sideset{}{^*}\sum_{\substack{\psi\ (\mathrm{mod}\ g\ell)\\ \psi\ne\overline{\chi}\\ \psi\ne\psi^*}}\tau(\overline{\psi})\sum_{d|kq}\delta(g\ell,kq,d,\psi)\\ &\times\sum_{sh=d}\sum_{\substack{sv\le M\\ (v,h)=1}}\overline{y}_{sv}\psi(v)\sum_{kqT/2\pi dv\le u\le kqT/\pi dv}(\Lambda_F*\chi)(hu)\psi(u).\end{aligned}\tag{6.58}$$

Since $R(x,h,k,\psi)=0$ for all $\psi\neq\overline{\chi},\psi^*$, we can bound $\mathcal{E}$ by

$$\mathcal{E}\ll\frac{\tau(\overline{\chi})}{q}\sum_{k\leq M}\frac{|\overline{x_k}|}{k}\sum_{\ell|q}\sum_{g|k}\sideset{}{^*}\sum_{\psi\pmod{g\ell}}|\tau(\overline{\psi})|\sum_{d|kq}|\delta(g\ell,kq,d,\psi)|$$
$$\times\sum_{sh=d}\sum_{\substack{sv\leq M\\(v,h)=1}}|\overline{y}_{sv}|\frac{qkT}{dv}h^{\theta_F}\tau(h)j(h)\log k(\log h)^3\exp(-c\sqrt{\log T}).$$

To estimate the error $\mathcal{E}$, we need the following lemmas on arithmetic functions.

Lemma 6.12. *For $d,k,q,\ell\in\mathbb{N}$, ψ a primitive character modulo $g\ell$ and $kq\ll T$ with $(k,q)=1$, we have the inequality*

$$\delta(g\ell,kq,d,\psi)\ll\frac{(d,kq/g\ell)}{\phi(kq)}.$$

Proof. From equation (6.31), we have

$$\delta(g\ell,kq,d,\psi)\ll\sum_{\substack{e|d\\e|kq/g\ell}}\frac{\phi(e)}{\phi(kq)}=\frac{(d,kq/g\ell)}{\phi(kq)}$$

since $1*\phi=id$. ◻

Lemma 6.13. *Let h be a positive multiplicative function and let $1\leq k,q\leq M$ and $(k,q)=1$. Then,*

$$\sum_{d|kq}\frac{(d,k)h(d)}{d}\ll(1*|h|)(k)\left\|\frac{h(n)}{n}\right\|_1.$$

Proof. Let $g=(d,k)$, and write $d=gd_1$ and $k=gk_1$. Then,

$$\sum_{d|kq}\frac{(d,kq)h(d)}{d}\ll\sum_{g|k}\sum_{d_1|kq/g}\frac{g}{gd_1}|h(gd_1)|\ll\sum_{g|k}|h(g)|\left\|\frac{h(n)}{n}\right\|_1=(1*|h|)(k)\left\|\frac{h(n)}{n}\right\|_1.$$

◻

Now we are ready to estimate $\mathcal{E}$. From Lemma 6.12 and $(k,q)=1$,

$$\delta(g\ell,kq,d,\psi)\ll\frac{(d,kq/g\ell)}{\phi(kq)}\ll\frac{(d,kq/g\ell)}{\phi(k)\phi(q)}.$$

We also have

$$\sum_{sh=d}\sum_{\substack{sv\leq M\\(v,h)=1}}\frac{|y_{sv}|}{v}\tau(h)j(h)\ll(\tau*|y|)(d)j(d)\left\|\frac{y_n}{n}\right\|_1.$$

Therefore, using $h, k \le qM \ll T$, we have

$$\begin{aligned}
\mathcal{E} \ll& \frac{|\tau(\overline{\chi})|}{q} \sum_{k\le M} \frac{|\overline{x_k}|}{k} \sum_{\ell | q} \sum_{g|k} \sideset{}{^*}\sum_{\psi \pmod{g\ell}} |\tau(\overline{\psi})| \sum_{d|kq} |\delta(g\ell, kq, d, \psi)| \\
&\times \sum_{sh=d} \sum_{\substack{sv\le M \\ (v,h)=1}} |\overline{y}_{sv}| \frac{qkT}{dv} h^{\theta_F} \tau(h) j(h) \log k (\log h)^3 \exp(-c\sqrt{\log T}) \\
\ll& \frac{(qM)^{\theta_F} q (\log M)^3 (\log q)^3 T}{\sqrt{q}} \sum_{k\le M} |x_k| \sum_{\ell|q} \sum_{g|k} \sqrt{g\ell}\phi(g\ell) \\
&\times \sum_{d|kq} \frac{(d, kq/g\ell)}{\phi(kq)} \frac{(\tau * |y|)(d) j(d)}{d} \left\| \frac{y_n}{n} \right\|_1 \exp(-c\sqrt{\log T}) \\
\ll& M^{\theta_F+\epsilon} q^{\frac12+\theta_F+\epsilon} T \sum_{\ell|q} \sum_{g\le M} \sum_{k\le M/g} |x_{gk}| (g\ell)^{3/2} \frac{(\tau_3 * |y|)(kq) j(k) j(q)}{\phi(gkq)} \left\| \frac{(\tau * |y|)(n)}{n} \right\|_1 \left\| \frac{y_n}{n} \right\|_1 \exp(-c\sqrt{\log T}) \\
\ll& M^{\theta_F+\epsilon} q^{1+\theta_F+\epsilon} T \sum_{g\le M} |x_g| \sqrt{g} \sum_{\substack{k\le M/g \\ (k,q)=1}} |x_k| \frac{(\tau_3 * |y|)(k) j(k)}{\phi(k)} \left\| \frac{(\tau * |y|)(n)}{n} \right\|_1 \left\| \frac{y_n}{n} \right\|_1 \exp(-c\sqrt{\log T}) \\
\ll& M^{\frac12+\theta_F+\epsilon} q^{1+\theta_F+\epsilon} \|y_n\|_\infty \|x_n\|_1 \left\| \frac{\tau_3 * |y|(n)}{n} \right\|_1 \left\| \frac{(\tau * |y|)(n)}{n} \right\|_1 \left\| \frac{y_n}{n} \right\|_1 \exp(-c\sqrt{\log T}). \qquad (6.59)
\end{aligned}$$

6.5.2. *Main Term Evaluation.* From (6.33), we have

$$\mathcal{M}_{\overline{\chi}} = \frac{\tau(\overline{\chi})}{q} \sum_{k\le M} \frac{\overline{x_k}}{k} \sum_{kqT/2\pi \le m \le kqT/\pi} a(m) \frac{\tau(\chi)\mu(\frac{k'}{q})\chi(\frac{k'}{q})\overline{\chi}(-m')}{\phi(k')} 1_{q|k'},$$

where $k' = \frac{kq}{(m,kq)}$ and $m' = \frac{m}{(m,kq)}$ and $q \mid k'$. Since x, y are supported on integers that are coprime to q, we have $(k,q) = 1$ and it follows that $(m,kq) = (m,k)(m,q)$. Thus if $q \mid k'$, we must have $(m,q) = 1$, so that $k' = qk/(m,k)$ and thus $m' = m/(m,k)$. We write $l = (m,k)$, $k = lk_1$, and $m = lm_1$, and then replace k_1 by k and m_1 by m. Using that $\tau(\chi)\tau(\overline{\chi})\chi(-1) = q$, we have

$$\begin{aligned}
\mathcal{M}_{\overline{\chi}} &= \sum_{k\le M} \frac{\overline{x_k}}{k} \sum_{\substack{kqT/2\pi \le m \le qkT/\pi \\ (m,q)=1}} a(m) \frac{1}{\phi(\frac{k}{(m,k)})\phi(q)} \mu\left(\frac{k}{(m,k)}\right) \chi\left(\frac{k}{(m,k)}\right) \overline{\chi}\left(\frac{m}{(m,k)}\right) \\
&= \sum_{lk\le M} \frac{\overline{x_{lk}}}{lk} \frac{\mu(k)\chi(k)}{\phi(k)} \sum_{\substack{kqT/2\pi \le m \le qkT/\pi \\ (m,kq)=1}} a(lm)\overline{\chi}(m) \\
&= -\frac{1}{\phi(q)} \sum_{lk\le M} \frac{\overline{x_{lk}}}{lk} \frac{\mu(k)\chi(k)}{\phi(k)} \sum_{sh=l} \sum_{\substack{sv\le M \\ (v,hkq)=1}} \overline{y}_{sv}\overline{\chi}(v) \sum_{\substack{kqT/2\pi \le uv \le kqT/\pi \\ (u,kq)=1}} \Lambda_F * \chi(hu)\overline{\chi}(u) \qquad (6.60)
\end{aligned}$$

After an application of Lemma 6.8 to (6.60), we obtain

$$\begin{aligned}
\mathcal{M}_{\overline{\chi}} = -& \frac{1}{\phi(q)} \sum_{lk\le M} \frac{\overline{x}_{lk}}{lk} \frac{\mu(k)\chi(k)}{\phi(k)} \sum_{sh=l} \sum_{\substack{sv\le M \\ (v,hkq)=1}} \overline{y}_{sv}\overline{\chi}(v) \\
&\times \frac{\phi(q)}{q} \Big\{ \frac{\phi(k)}{k} \frac{kqT}{2\pi v} \left(\chi(h) \log\left(\frac{2kqT}{\pi v e}\right) f_{-1} + X_1(h,kq) + \chi(h) X_2(kq) \right) \\
&+ O\left(h^{\theta_F} \tau(h) j(h) (\log h)^3 j(kq) \log(kq) \frac{kqT}{v} \exp(-c\sqrt{\log T}) \right) \Big\}. \qquad (6.61)
\end{aligned}$$

Let $\mathcal{E}_{\overline{\chi}}$ denote the contribution of the O-terms in (6.61). After replacing l by sh and using $h \le qM$, we have

$$\begin{aligned}\mathcal{E}_{\overline{\chi}} \ll & q^{\theta_F+\epsilon}M^{\theta_F+\epsilon}T \sum_{shk\le M} \frac{|\overline{x}_{shk}|}{shk}\frac{k}{\phi(k)}\tau(h)j(h)\sum_{sv\le M}\frac{|\overline{y}_{sv}|}{v}\exp(-c\sqrt{\log T})\\ \ll & q^{\theta_F+\epsilon}M^{\theta_F+\epsilon}T \left\|\frac{|x_s y_s|}{s}\right\|_1 \left\|\frac{(\tau_3 * |x|)(n)}{n}\right\|_1 \left\|\frac{y_v}{v}\right\|_1 \exp(-c\sqrt{\log T}).\end{aligned}$$

Since x_k are supported on integers k with $(k,q)=1$, thus $l \nmid q$ and $h \nmid q$. Upon writing $l = sh$ and $hk = u$, we find that

$$\begin{aligned}\mathcal{M}_{\overline{\chi}} =& -\frac{T}{2\pi}\sum_{su\le M}\frac{\overline{x}_{su}\chi(u)}{su}\sum_{\substack{sv\le M\\(v,uq)=1}}\frac{\overline{y}_{sv}\overline{\chi}(v)}{v}\sum_{hk=u}\mu(k)\left(\log\left(\frac{2kqT}{\pi ve}\right)f_{-1}+\widetilde{X_1}(h,k)+\widetilde{X_2}(k)\right)+\mathcal{E}_{\overline{\chi}}\\ =& -\frac{T}{2\pi}\sum_{su\le M}\frac{\overline{x}_{su}\chi(u)}{su}\sum_{\substack{sv\le M\\(v,uq)=1}}\frac{\overline{y}_{sv}\overline{\chi}(v)}{v}\sum_{hk=u}\mu(k)\left(\log\left(\frac{2qT}{\pi ve}\right)f_{-1}+f_{-1}\log k\right)\\ & -\frac{T}{2\pi}\sum_{su\le M}\frac{\overline{x}_{su}\chi(u)}{su}\sum_{\substack{sv\le M\\(v,uq)=1}}\frac{\overline{y}_{sv}\overline{\chi}(v)}{v}\sum_{hk=u}\mu(k)\left(\widetilde{X_1}(h,k)+f_{-1}\widetilde{X_2}(k)\right)+\mathcal{E}_{\overline{\chi}}\\ =& -\frac{T}{2\pi}\sum_{su\le M}\frac{\overline{x}_{su}\chi(u)}{su}\sum_{\substack{sv\le M\\(v,uq)=1}}\frac{\overline{y}_{sv}\overline{\chi}(v)}{v}\left(\delta(u)\log\left(\frac{2qT}{\pi ve}\right)f_{-1}-\Lambda(u)f_{-1}\right)\\ & -\frac{T}{2\pi}\sum_{su\le M}\frac{\overline{x}_{su}\chi(u)}{su}\sum_{\substack{sv\le M\\(v,uq)=1}}\frac{\overline{y}_{sv}\overline{\chi}(v)}{v}\sum_{hk=u}\mu(k)\left(\widetilde{X_1}(h,k)+f_{-1}\widetilde{X_2}(k)\right)+\mathcal{E}_{\overline{\chi}},\end{aligned}\tag{6.62}$$

where

$$\begin{aligned}\delta(u) &= \begin{cases}1, & \text{if } u=1,\\ 0, & \text{if } u>1,\end{cases}\\ \widetilde{X_1}(h,k) &= \sum_{a|(h,k)}\Lambda_F(a)\overline{\chi}(a)+\sum_{p^l|h,p\nmid k}\overline{\chi}(p^l)\eta(1;p,l,\overline{\chi})(1-p^{-1}),\\ \widetilde{X_2}(k) &= f_0-\eta(1;kq,\overline{\chi})+\Big(\gamma+\sum_{p|kq}\frac{\log p}{p-1}\Big)f_{-1}.\end{aligned}$$

For $\mathcal{M}_{\psi^*}$, we write the modulus of ψ^* as $g^* = g_0\ell_0$, where $\ell_0 = (g^*,q)$ and $(g_0,q)=1$.

$$\mathcal{M}_{\psi^*} = \frac{\tau(\overline{\chi})}{q}\sum_{k\le M}\frac{\overline{x}_k}{k}\sum_{kqT/2\pi<m\le kqT/\pi}a(m)\frac{1}{\phi(k')}\mu\left(\frac{k'}{g_0\ell_0}\right)\overline{\psi^*}\left(\frac{k'}{g_0\ell_0}\right)\tau(\overline{\psi^*})\psi^*(-m')1_{g_0\ell_0|k'},$$

where $k' = \frac{kq}{(kq,m)}, m' = \frac{m}{(kq,m)}$ and $g_0\ell_0 \mid k'$. We further choose x, y supported on integers that are coprime to qg_0. If $g_0\ell_0 \mid k' \mid kq$, we must have $g_0 \mid k$ since $(k,q)=1, (g_0,q)=1$. From the support of x_k we have $(k,g_0)=1$ and thus we must have $g_0 = 1$. Therefore, we only need to consider the case when ψ^* is a character modulo ℓ_0 with $\ell_0 \mid q$. Since $(k,q)=1$, we see that $\ell_0 \mid k'$ implies that $\ell_0 \mid \frac{q}{(q,m)}$. If we write $l=(m,k)$, $\ell=(q,m)$, $k=lk_1$, $m=\ell l m_1, q=\ell\ell_0 q_1$, then $(m_1,k_1)=1, (m_1,q_1\ell_0)=1$ and

$\ell_0 \mid q/\ell$. Thus,

$$\begin{aligned}
\mathcal{M}_{\psi^*} &= \frac{\tau(\overline{\chi})}{q} \sum_{\ell|q/\ell_0} \sum_{l\le M} \sum_{k_1\le M/l} \frac{\overline{x}_{lk_1}}{lk_1} \frac{1}{\phi(\ell_0 k_1 q_1)} \mu(k_1 q_1)\overline{\psi^*}(k_1 q_1)\tau(\overline{\psi^*}) \\
&\quad\times \sum_{\substack{\ell_0 k_1 q_1 T/2\pi < m_1 \le \ell_0 k_1 q_1 T/\pi \\ (m_1, \ell_0 k_1 q_1)=1}} a(\ell l m_1)\psi^*(-m_1) \\
&= -\frac{\tau(\overline{\chi})}{q} \sum_{\ell|q/\ell_0} \sum_{l\le M} \sum_{k_1\le M/l} \frac{\overline{x}_{lk_1}}{lk_1} \frac{1}{\phi(\ell_0 k_1 q_1)} \mu(k_1 q_1)\overline{\psi^*}(k_1 q_1)\tau(\overline{\psi^*})\psi^*(-1) \\
&\quad\times \sum_{sh=\ell l} \sum_{\substack{sv\le M \\ (v,h\ell_0 k_1 q_1)=1}} \overline{y}_{sv}\psi^*(v) \sum_{\substack{k_1 qT/2\pi\ell < uv \le k_1 qT/\pi\ell \\ (u,\ell_0 k_1 q_1)=1}} (\Lambda_F * \chi)(hu)\psi^*(u) \\
&= -\frac{\tau(\overline{\chi})}{q} \sum_{\ell|q/\ell_0} \sum_{l\le M} \sum_{k_1\le M/l} \frac{\overline{x}_{lk_1}}{lk_1} \frac{1}{\phi(k_1 q/\ell)} \mu(k_1 \frac{q}{\ell\ell_0})\overline{\psi^*}(k_1 \frac{q}{\ell\ell_0})\overline{\tau(\psi^*)} \\
&\quad\times \sum_{sh=\ell l} \sum_{\substack{sv\le M \\ (v,hk_1 q/\ell)=1}} \overline{y}_{sv}\psi^*(v)\Big\{ \frac{k_1 qT}{2\pi\ell v}\Phi(1; k_1 q/\ell, \chi\psi^*)L(1,\chi\psi^*)\chi(h)\tilde{f}_{-1} \\
&\quad + O\left(h^{\theta_F}\tau(h)j(h)(\log h)^3 j(qk)\log(qk)\frac{k_1 qT}{\ell v}\exp(-c\sqrt{\log T})\right) \Big\},
\end{aligned} \tag{6.63}$$

where $\tilde{f}_{-1} = \mathrm{Res}_{z=1}\, G(z,\psi^*)$. Let $\mathcal{E}_{\psi^*}$ denote the contribution of O-terms in (6.63). Since $(sl,q) = 1, \ell \mid q, sh = \ell l$, we must have $\ell \mid h$. After replacing sh by ℓl and h by ℓh, we obtain a bound for $\mathcal{E}_{\psi^*}$ as

$$\begin{aligned}
\mathcal{E}_{\psi^*} &\ll q^{\theta_F+\epsilon}M^{\theta_F+\epsilon} \sum_{\ell|q/\ell_0} \sum_{l\le M} \sum_{k_1\le M/l} \frac{|\overline{x}_{lk_1}|}{lk_1} \frac{1}{\phi(\ell_0 k_1 q_1)} \sum_{sh=\ell l} \sum_{\substack{sv\le M \\ (v,h\ell_0 k_1 q_1)=1}} |\overline{y}_{sv}| \frac{k_1 qT}{\ell v}\exp(-c\sqrt{\log T}) \\
&\ll q^{\theta_F+\epsilon}M^{\theta_F+\epsilon} T \sum_{\ell|q/\ell_0} \sum_{sh\le M} \sum_{k_1\le M/sh} \frac{|\overline{x}_{shk_1}|}{shk_1} \sum_{v\le M/s} \frac{|\overline{y}_{sv}|}{v}\tau(h)j(h)\exp(-c\sqrt{\log T}) \\
&\ll q^{\theta_F+\epsilon}M^{\theta_F+\epsilon} T \left\| \frac{|\overline{x}_s \overline{y}_s|}{s} \right\|_1 \left\| \frac{(\tau_3 * |\overline{x}|)(n)}{n} \right\|_1 \left\| \frac{\overline{y}_v}{v} \right\|_1 \exp\left(-c\sqrt{\log T}\right).
\end{aligned} \tag{6.64}$$

Now we further require that x, y be supported on squarefree integers whose prime factors are $\equiv 1 \bmod qg_0$. It follows that if $x_{k_1} \neq 0$, then $\chi(k_1) = \psi^*(k_1) = 1$ and $(k_1, qg_0) = 1$, thus

$$\Phi(z; \ell_0 k_1 q_1, \chi\psi^*) = \prod_{p|\ell_0 k_1 q_1} (1 - \chi\psi^*(p)p^{-1}) = \prod_{\substack{p|k_1 \\ p\nmid g_0}} (1 - p^{-1}) = \frac{\phi(k_1)}{k_1}.$$

Using this in (6.63), we simplify $\mathcal{M}_{\psi^*}$ as

$$\begin{aligned}
\mathcal{M}_{\psi^*} = -\frac{\tau(\overline{\chi})}{q} \sum_{\ell|q/\ell_0} \sum_{l\le M} \sum_{k_1\le M/l} \frac{\overline{x}_{lk_1}}{lk_1} \frac{1}{\phi(k_1 q/\ell)} \mu\left(k_1 \frac{q}{\ell\ell_0}\right) \overline{\psi^*}\left(k_1 \frac{q}{\ell\ell_0}\right) \overline{\tau(\psi^*)} \sum_{sh=\ell l} \sum_{\substack{sv\le M \\ (v,hk_1 q/\ell)=1}} \overline{y}_{sv}\psi^*(v) \\
\times \frac{k_1 qT}{2\pi\ell v} \frac{\phi(k_1)}{k_1} L(1,\chi\psi^*)\chi(h)\tilde{f}_{-1} + \mathcal{E}_{\psi^*},
\end{aligned}$$

where $\mathcal{E}_{\psi^*}$ is estimated by (6.64). Now we replace ℓl by sh, since $(k_1, q) = 1$, the main term in $\mathcal{M}_{\psi^*}$ becomes

$$-\tau(\overline{\chi})\overline{\tau(\psi^*)} \sum_{\ell|q/\ell_0} \sum_{sh\le M} \sum_{k_1\le M\ell/sh} \frac{\overline{x}_{shk_1/\ell}}{shk_1/\ell} \frac{\mu(k_1 \frac{q}{\ell\ell_0})\overline{\psi^*}(k_1 \frac{q}{\ell\ell_0})}{\phi(q/\ell)} \sum_{\substack{sv\le M \\ (v,hk_1 q/\ell)=1}} \frac{\overline{y}_{sv}}{v}\psi^*(v)\frac{T}{2\pi\ell} L(1,\chi\psi^*)\chi(h)\tilde{f}_{-1}.$$

Since $(sk_1, q) = 1$, we must have $\ell \mid h$. After replacing h by ℓh, the above becomes

$$-\tau(\overline{\chi})\overline{\tau(\psi^*)} \sum_{\ell | q/\ell_0} \sum_{sh \leq M/\ell} \sum_{k_1 \leq M\ell/sh} \frac{\overline{x}_{shk_1}}{shk_1} \frac{\mu(k_1 \frac{q}{\ell\ell_0})\overline{\psi^*}(k_1 \frac{q}{\ell\ell_0})}{\phi(q/\ell)} \sum_{\substack{sv \leq M \\ (v, hk_1 q)=1}} \frac{\overline{y}_{sv}}{v} \psi^*(v) \frac{T}{2\pi\ell} L(1, \chi\psi^*)\chi(h\ell)\tilde{f}_{-1}.$$

Since $\ell \mid q$, the non zero contribution comes only from $\ell = 1$. By the assumption on the support of x_n, we have $\chi(h) = 1$ if $x_h \neq 0$. After writing $hk_1 = u$, we see that the main term in $\mathcal{M}_{\psi^*}$ is simplified to

$$\begin{aligned}
&-\tau(\overline{\chi})\overline{\tau(\psi^*)} \sum_{sh \leq M} \sum_{k_1 \leq M/sh} \frac{\overline{x}_{shk_1}}{shk_1} \frac{\mu(k_1 q/\ell_0)\overline{\psi^*}(k_1 q/\ell_0)}{\phi(q)} \sum_{\substack{sv \leq M \\ (v, hk_1 q)=1}} \frac{\overline{y}_{sv}}{v} \psi^*(v) \frac{T}{2\pi} L(1, \chi\psi^*)\tilde{f}_{-1} \\
&= -\frac{\tau(\overline{\chi})\overline{\tau(\psi^*)}T}{2\pi\phi(q)} \sum_{s \leq M} \sum_{u \leq M/s} \frac{\overline{x}_{su}}{su} \sum_{k_1 | u} \mu(k_1)\mu(q/\ell_0)\overline{\psi^*}(k_1)\overline{\psi^*}(q/\ell_0) \sum_{\substack{sv \leq M \\ (v, uq)=1}} \frac{\overline{y}_{sv}}{v} \psi^*(v) L(1, \chi\psi^*)\tilde{f}_{-1} \\
&= -\frac{\tau(\overline{\chi})\overline{\tau(\psi^*)}T}{2\pi\phi(q)} \mu(q/\ell_0)\overline{\psi^*}(q/\ell_0) \sum_{s \leq M} \sum_{u \leq M/s} \frac{\overline{x}_{su}}{su} \sum_{k_1 | u} \mu(k_1) \sum_{\substack{sv \leq M \\ (v, uq)=1}} \frac{\overline{y}_{sv}}{v} \psi^*(v) L(1, \chi\psi^*)\tilde{f}_{-1} \\
&= -\frac{\tau(\overline{\chi})\overline{\tau(\psi^*)}T}{2\pi\phi(q)} \mu(q/\ell_0)\overline{\psi^*}(q/\ell_0) \sum_{s \leq M} \sum_{u \leq M/s} \frac{\overline{x}_s}{s} \sum_{\substack{sv \leq M \\ (v, q)=1}} \frac{\overline{y}_{sv}}{v} L(1, \chi\psi^*)\tilde{f}_{-1}.
\end{aligned}$$

Therefore,

$$\mathcal{M}_{\psi^*} = -\frac{\tau(\overline{\chi})\overline{\tau(\psi^*)}T}{2\pi} \frac{\mu(q/\ell_0)\overline{\psi^*}(q/\ell_0)}{\phi(q)} L(1, \chi\psi^*)\tilde{f}_{-1} \sum_{s \leq M} \frac{\overline{x}_s}{s} \sum_{\substack{sv \leq M \\ (v, q)=1}} \frac{\overline{y}_{sv}}{v} + \mathcal{E}_{\psi^*}. \tag{6.65}$$

If there exist some positive constant $a = a(\chi, F)$ such that both F_ψ and $L(s, \chi\psi)$ have no zeros in the region $\Re(s) \geq 1 - a$ for all ψ, then we can replace the term $\exp(-c\sqrt{\log T})$ in the error terms $\mathcal{E}, \mathcal{E}'$ can be replaced by $T^{-\delta+\epsilon}$ for some small enough $\delta = \delta(F, a)$, since ψ has conductor $\leq M \leq \sqrt{T}$.

7. Resonator Coefficients

In this section, we define the resonator coefficients and give their properties.

Let χ be a Dirichlet character and denote ψ^* (mod g^*) as the Dirichlet character $\neq \overline{\chi}$ such that $F_{\psi^*}(s)$ has a pole at $s = 1$. If no such character exists, set $g^* = 1$. Let $f(n)$ be multiplicative and supported on squarefree integers $n \leq M$. Let $d = \mathrm{lcm}(q, g^*)$, $K = \sqrt{\phi(d) \log M \log_2 M}$, and B_M be as in (iii) in the definition of $\mathcal{S}^*$. Define

$$f(p) = \frac{K}{\log p}, \text{ if } K^2 \leq p \leq \exp((\log K)^2), p \neq B_M,\ p \equiv 1 \pmod{d}, \tag{7.1}$$

and 0 otherwise. Then, we have the following estimates of norms involving f.

Lemma 7.1. *Let $f(n)$ be defined by (7.1). Then for M sufficiently large, we have*

(1) $|f(n)| \leq n^{1/2}$,

(2) $\left\| \frac{f(n)}{n} \right\|_1 \ll \exp\left(\frac{1}{\phi(d)}(1 + o(1)) \frac{K}{\log K^2} \right)$,

(3) $\|f(n)\|_1 \ll M \exp\left(\frac{1}{\phi(d)}(1 + o(1)) \frac{K}{\log K^2} \right)$,

(4) $\left\| \frac{f(n)^2}{n} \right\|_1 \ll \exp\left(\frac{1}{2\phi(d)}(1 + o(1)) \frac{K^2}{(\log K^2)^2} \right)$,

(5) $\|f(n)^2\|_1 \ll M \exp\left(\frac{1}{2\phi(d)}(1 + o(1)) \frac{K^2}{(\log K^2)^2} \right)$,

(6) $\|j(n)(\tau_r * f)(n)f(n)/n\|_1 \ll \exp\left(\frac{1}{2\phi(d)}(1 + o(1)) \frac{K^2}{(\log K^2)^2} \right)$,

where $j(n)$ is defined in (6.36).

Proof. (1) Since $K^2 \le p$, we see that $f(p) \le \sqrt{p}\frac{K}{\sqrt{p}\log p} \le \sqrt{p}$. Thus $f(n) \le \sqrt{n}$ by multiplicativity.

(2) Using the multiplicativity of $f(n)$, we have an upper bound

$$\|f(n)/n\|_1 = \sum_{n=1}^{M} \frac{f(n)}{n} \le \prod_p \left(1 + \frac{f(p)}{p}\right) \le \exp\left(\sum_p \frac{f(p)}{p}\right).$$

From the prime number theorem in an arithmetic progression,

$$\sum_p \frac{f(p)}{p} \le \sum_{\substack{K^2 \le p \le \exp((\log K)^2) \\ p \equiv 1 \pmod d}} \frac{K}{p \log p} = (1+o(1))\frac{1}{\phi(d)} \int_{K^2}^{\exp((\log K)^2)} \frac{K}{x \log^2 x} dx$$
$$= (1+o(1))\frac{1}{\phi(d)}\frac{K}{\log K^2}.$$

(3) Using $n \le M$, Rankin's trick and part (2), we have

$$\|f(n)\|_1 = \sum_{n=1}^{M} f(n) \le M \sum_{n=1}^{M} \frac{f(n)}{n} \le M \left\|\frac{f(n)}{n}\right\|_1 \ll M \exp\left(\frac{(1+o(1))}{\phi(d)}\frac{K}{\log K^2}\right),$$

as $M \to \infty$.

(4) Similarly to (2), we have

$$\sum_{n=1}^{M} \frac{f(n)^2}{n} \le \prod_p \left(1 + \frac{f(p)^2}{p}\right) \le \exp\left(\sum_p \frac{f(p)^2}{p}\right).$$

Using the prime number theorem in an arithmetic progression again, we derive that

$$\sum_{K \le p \le \exp((\log K)^2)} \frac{f(p)^2}{p} \le \sum_{\substack{K^2 \le p \le \exp((\log K)^2) \\ p \equiv 1 \pmod d}} \frac{K^2}{p \log^2 p}$$
$$= \frac{1}{\phi(d)}(1+o(1)) \int_{K^2}^{\exp((\log K)^2)} \frac{K^2}{x \log^3 x} dx$$
$$= (1+o(1))\frac{1}{2\phi(d)}\frac{K^2}{(\log K^2)^2},$$

as $M \to \infty$.

(5) Similarly to part (3), we have $\|f^2\|_1 \le M \left\|\frac{f^2(n)}{n}\right\|_1$.

(6) Note that since n is squarefree and $j(n)$ and $\tau_k(n)$ are multiplicative,

$$\sum_{n \le M} \frac{j(n)(\tau_k * f)(n) f(n)}{n} \le \prod_p \left(1 + \frac{j(p)(\tau_k * f)(p) f(p)}{p}\right).$$

Since $j(p) = 1 + O(\sqrt{p})$, $f(p) = \frac{K}{\log p}$, and $(\tau_k * f)(p) = \frac{K}{\log p} + k$, we obtain

$$\sum_{\substack{K^2 \le p \le \exp((\log K)^2) \\ p \equiv 1 \mod d}} \frac{j(p)(\tau_k * f)(p) f(p)}{p}$$
$$\le \sum_{\substack{K^2 \le p \le \exp((\log K)^2) \\ p \equiv 1 \mod d}} \left(\frac{K^2}{p \log^2 p} + O\left(\frac{K}{p \log p} + \frac{K^2}{p^{3/2-\epsilon}}\right)\right)$$
$$= \frac{1}{2\phi(d)}(1+o(1))\frac{K^2}{(\log K^2)^2}, \quad M \to \infty.$$

□

Lemma 7.2. *Let $f(p)$ be defined in (7.1). Set*

$$\mathcal{Q}_0 = \prod_p \left(1 + \frac{f(p)^2}{p}\right), \ \ \mathcal{Q}_1 = \prod_p \left(1 + \frac{f(p)^2}{p} + \frac{f(p)}{p}\right). \tag{7.2}$$

Then, as $M \to \infty$,

(1) $\displaystyle\sum_{nu \le M} \frac{f(u)f(nu)}{nu} = \mathcal{Q}_1(1 + o(1))$,

(2) $\displaystyle\sum_{n \le M} \frac{f(n)^2}{n} \le \mathcal{Q}_0$,

(3) $\displaystyle\frac{\mathcal{Q}_1}{\mathcal{Q}_0} = \exp\left(\frac{1}{\phi(d)} \frac{K}{\log K^2}(1 + o(1))\right)$,

(4) $\displaystyle\sum_{n \le M} \frac{|\Lambda_F(n)| f(n)}{n(1 + f(n)^2 n^{-1})} \ll K \log_2 M$,

(5) $\displaystyle\sum_{n \le M} \sum_{m=1}^{\infty} \frac{|\Lambda_F(p^m)| f(p)}{p^m(1 + f(p)^2 p^{-1})} \ll K \log_2 M$,

(6) $\displaystyle\sum_{nu \le M} \frac{\log n f(u) f(nu)}{nu} \ll \mathcal{Q}_1 K \log_2 M$.

Proof. (1) Since f is multiplicative and supported on squarefree numbers,

$$\begin{aligned}\sum_{nu \le M} \frac{f(n)f(nu)}{nu} &= \sum_{n \le M} \frac{f(n)}{n} \sum_{\substack{u \le M/n \\ (u,n)=1}} \frac{f(u)^2}{u} \\ &= \sum_{n \le M} \frac{f(n)}{n} \left(\prod_{(p,n)=1} \left(1 + \frac{f(p)^2}{p}\right) - \sum_{\substack{u \ge M/n \\ (n,u)=1}} \frac{f(u)^2}{u}\right).\end{aligned}$$

By Rankin's trick, the contribution from $u > M/n$ is bounded by

$$\sum_{n \le M} \frac{f(n)}{n} \left(\frac{n}{M}\right)^{\alpha} \sum_{\substack{u=1 \\ (u,n)=1}}^{\infty} \frac{f(u)^2 u^{\alpha}}{u} \le \frac{1}{M^{\alpha}} \prod_p \left(1 + f(p)^2 p^{\alpha - 1} + f(p) p^{\alpha - 1}\right) \tag{7.3}$$

for any $\alpha > 0$. By Rankin's trick again, the main term becomes

$$\prod_p \left(1 + \frac{f(p)^2}{p} + \frac{f(p)}{p}\right) + O\left(\frac{1}{M^{\alpha}} \prod_p \left(1 + \frac{f(p)^2}{p} + \frac{f(p)p^{\alpha}}{p}\right)\right). \tag{7.4}$$

Combining (7.4) and (7.3), we deduce that

$$\sum_{nu \le M} \frac{f(u)f(nu)}{nu} = \mathcal{Q}_1 + O\left(\frac{1}{M^{\alpha}} \prod_p \left(1 + f(p)^2 p^{\alpha - 1} + f(p) p^{\alpha - 1}\right)\right).$$

Taking $\alpha = 1/(\log K)^3$, we see that the ratio of the error to the main term is bounded by

$$\begin{aligned}
&\ll \exp\left(-\alpha \log M + \sum_{\substack{K^2 \le p \le \exp((\log K)^2) \\ p \equiv 1 \pmod d}} (p^\alpha - 1)\left(\frac{K^2}{p\log^2 p} + \frac{K}{p\log p}\right)\right) \\
&\ll \exp\left(-\alpha \log M + \alpha \frac{K^2}{\phi(d)\log K^2} - \alpha \frac{K^2}{\phi(d)(\log K)^2}\right) \qquad (7.5) \\
&\ll \exp\left(-\alpha \frac{\log M}{\log_2 M}\right).
\end{aligned}$$

Note that we used the fact that $\frac{K^2}{\phi(d)\log K^2} \le \log M$ in the last step to ensure (7.5) is $o(1)$. Therefore,

$$\sum_{nu\le M} \frac{f(u)f(nu)}{nu} = \mathcal{Q}_1(1+o(1)).$$

(2) We have the inequality

$$\sum_{n\le M} \frac{f(n)^2}{n} \le \sum_{n} \frac{f(n)^2}{n} = \prod_p \left(1 + \frac{f(p)^2}{p}\right) = \mathcal{Q}_0.$$

(3) From the definitions of $\mathcal{Q}_0$ and $\mathcal{Q}_1$ defined in (7.2), it can be seen that

$$\frac{\mathcal{Q}_1}{\mathcal{Q}_0} = \prod_p \left(1 + \frac{f(p)}{p(1+f(p)^2p^{-1})}\right).$$

From (iii), we have $B_M \gg \log_2 M \gg \log K$. Since $\log K \ll \log_2 T$, we have

$$\frac{K}{B_M \log B_M} \ll \frac{K}{\log K \log_2 K} = o\left(\frac{K}{\log K^2}\right).$$

It follows that

$$\begin{aligned}
\sum_{K\le p\le \exp((\log K)^2)} \frac{f(p)}{p(1+f(p)^2p^{-1})} &= \sum_{\substack{K^2 \le p \le \exp((\log K)^2) \\ p\equiv 1 \mod d}} \frac{K}{p\log p}(1+o(1)) + o(\frac{K}{\log K^2}) \\
&= (1+o(1))\frac{1}{\phi(d)}\frac{K}{\log K^2}.
\end{aligned}$$

(4) Since f is supported on squarefree integers, and $\Lambda_F(n)$ is supported on prime powers,

$$\begin{aligned}
\sum_{n\le M} \frac{|\Lambda_F(n)|f(n)}{n(1+f(n)^2p^{-1})} &= K\sum_{p\le M} \frac{|\lambda_F(p)|}{p(1+f(p)^2p^{-1})} \\
&\ll K\left(\sum_{p\le M}\frac{1}{p}\right)^{1/2}\left(\sum_{p\le M}\frac{|\lambda_F(p)|^2}{p}\right)^{1/2} \\
&\ll K\log_2 M,
\end{aligned}$$

where the last inequality follows from (3.6) and partial summation.

(5) Since we have $\Lambda_F(n) \ll n^{1/2}$, we see that

$$\sum_p \sum_{m=4}^{\infty} \frac{|\Lambda_F(p^m)|f(p)}{p^m} \ll K\sum_p \frac{1}{p^2}\frac{1}{1-p^{-1/2}} \ll K.$$

This shows that

$$\begin{aligned}
\sum_{p\le M}\sum_{m=1}^{\infty}\frac{|\Lambda_F(p^m)|f(p)}{p^m} &\ll \sum_{p\le M}\sum_{m\le 4}\frac{|\Lambda_F(p^m)|f(p)}{p}+K\\
&\ll K\sum_{n\le M^4}\frac{|\lambda_F(n)|}{n}+K\ll K\log_2 M\\
&\ll K\left(\sum_{n\le M^4}\frac{|\lambda_F(n)|^2}{n}\right)^{1/2}\left(\sum_{p}\sum_{m\le 4}\frac{1}{p^m}\right)^{1/2}\\
&\ll K\log_2 M.
\end{aligned}$$

(6) From the identity $\log n=\sum_{d|n}\Lambda(d)$,

$$\begin{aligned}
\sum_{nu\le M}\frac{\log n f(u)f(nu)}{nu} &= \sum_{k\le M}\frac{\Lambda(k)f(k)}{k}\sum_{\substack{nu\le M/k\\(nu,k)=1}}\frac{f(u)f(nu)}{nu}\\
&\le \sum_{k\le M}\frac{\Lambda(k)f(k)}{k}\sum_{\substack{n\le M/k\\(n,k)=1}}\frac{f(n)}{n}\prod_{(p,nk)=1}\left(1+\frac{f(p)^2}{p}\right)\\
&\le \mathcal{Q}_0\sum_{k\le M}\frac{\Lambda(k)f(k)}{k}\sum_{\substack{n\le M\\(n,k)=1}}\frac{f(n)}{n}\prod_{p|nk}\left(1+\frac{f(p)^2}{p}\right)^{-1}\\
&\le \mathcal{Q}_0\sum_{n\le M}\frac{f(n)}{n}\prod_{p|n}\left(1+\frac{f(p)^2}{n}\right)^{-1}\sum_{k\le M}\frac{\Lambda(k)f(k)}{k}\prod_{p|k}\left(1+\frac{f(p)^2}{p}\right)^{-1}\\
&\le \mathcal{Q}_1 K\log_2 M.
\end{aligned}$$

□

8. Proof of Theorem 1.1-1.4

Now we are ready to prove Theorem 1.1-Theorem 1.4. Let $F\in\mathcal{S}^*$ and $\psi^* \pmod{g^*}$ be the Dirichlet character $\neq\overline{\chi}$ such that $F_{\psi^*}(s)$ has a pole at $s=1$. Let $\chi\neq\psi^*$ be a Dirichlet character. Define $x_n=y_n=f(n)$, where $f(n), n=1,\dots,M$ are the resonator coefficients defined in Section 7. Using Theorem 6.1 and Lemma 7.1, we see that the error term becomes $o(T)$ upon taking $M=\exp(c'\sqrt{\log T})$ for some $c'>0$. If there exist positive constants a depending on F, and χ such that all three of $L(s,\chi\psi)$, F, and F_ψ have no zeros in the region $\Re(s)\ge 1-a$ for all ψ, then we can take $M=T^{\delta'}$ for some small positive constant $\delta'=\delta'(F,\chi)$. Thus it remains to compute the main terms. Let g be the multiplicative function supported on squarefree integers with $g(p)=1+\frac{f(p)^2}{p}$. From the multiplicativity of f and g, we have

$$\begin{aligned}
\sum_{nu\le M}\frac{x_u y_{nu}}{nu}(\Lambda_F*1)(n) &= \sum_{nu\le M}\frac{f(u)f(nu)(\Lambda_F*1)(n)}{nu}\\
&= \sum_{k\le M}\frac{\Lambda_F(k)f(k)}{k}\sum_{\substack{nu\le M/k\\(nu,k)=1}}\frac{f(u)f(nu)}{nu}\\
&\le \sum_{k\le M}\frac{\Lambda_F(k)f(k)}{k}\sum_{\substack{n\le M/k\\(n,k)=1}}\frac{f(n)}{n}\prod_{(p,nk)=1}\left(1+\frac{f(p)^2}{p}\right)\\
&\le \mathcal{Q}_0\sum_{k\le M}\frac{\Lambda_F(k)f(k)}{k}\sum_{\substack{n\le M\\(n,k)=1}}\frac{f(n)}{n}\prod_{p|nk}\left(1+\frac{f(p)^2}{p}\right)^{-1}\\
&\le \mathcal{Q}_0\sum_{n\le M}\frac{f(n)}{ng(n)}\sum_{k\le M}\frac{\Lambda_F(k)f(k)}{kg(k)}\\
&\le \mathcal{Q}_1\sum_{p\le M}\frac{\Lambda_F(p)f(p)}{pg(p)} \ll \mathcal{Q}_1 K\log_2 M, \qquad (8.1)
\end{aligned}$$

where we used (2) and (4) from Lemma 7.2. Next, we give an upper bound for the terms involving $r_3(u)$. From the assumption that $F_{\overline{\chi}}$ has no pole at $z=1$, we see that $f_{-1}=0$, and thus

$$\begin{aligned}
r_3(u)=r_4(u) &= \sum_{hk=u}\mu(k)\left(f_0-\sum_{p|k}\sum_{m=1}^{\infty}\frac{\Lambda_F(p^m)}{p^m}+\sum_{p|(h,k)}\Lambda_F(a)+\sum_{p|h,p\nmid k}\sum_{m=1}^{\infty}\frac{\Lambda_F(p^m)}{p^m}(p-1)\right)\\
&= \left(f_0-\sum_{p|u}\sum_{m=1}^{\infty}\frac{\Lambda_F(p^m)}{p^m}\right)\sum_{k|u}\mu(k)+\sum_{hk=u}\mu(k)\sum_{p|h}\sum_{m=1}^{\infty}\frac{\Lambda_F(p^m)}{p^{m-1}}\\
&=: f_0\delta(u)-\sum_{p|u}\sum_{m=1}^{\infty}\frac{\Lambda_F(p^m)}{p^{m-1}}\sum_{k|\frac{u}{p}}\mu(k),
\end{aligned}$$

which is non zero only when $u=1$ or u is a prime. Thus,

$$\begin{aligned}
\sum_{u\le M}\sum_{\substack{v\le M\\(v,u)=1}}\frac{x_u y_v r_3(u)}{uv}\sum_{\substack{s\le M\\(s,uv)=1}}\frac{x_s y_s}{s} &\le \mathcal{Q}_0\sum_{u\le M}\sum_{\substack{v\le M\\(v,u)=1}}\frac{x_u y_v r_3(u)}{uvg(uv)}\\
&\le \mathcal{Q}_0\sum_{v\le M}\frac{y_v}{vg(v)}\sum_{\substack{u\le M\\(u,v)=1}}\frac{x_u}{ug(u)}r_3(u)\\
&\ll \mathcal{Q}_1\sum_{p\le M}\frac{x_p}{pg(p)}\sum_{m=1}^{\infty}\frac{\Lambda(p)|\lambda_F(p^m)|}{p^{m-1}}\\
&\le \mathcal{Q}_1 K\log_2 M. \qquad (8.2)
\end{aligned}$$

where the last inequality follows from (5) of Lemma 7.2. Therefore, from Theorem 6.1, (8.1) and (8.2), we have

$$S_1=\frac{T}{2\pi}d_F P_1(\log(\lambda Q^2)^{1/d_F}T)(1+o(1))\mathcal{Q}_1+O\left(T\mathcal{Q}_1 K\log_2 M\right).$$

Next we consider S_0 (cf. (2.2)). Using Lemma 7.2, we have that

$$\sum_{nu\le M}\frac{\Lambda_F(n)f(n)f(nu)}{nu}=\sum_{n\le M}\frac{\Lambda_F(n)f(n)}{n}\sum_{\substack{u\le M/n\\(u,n)=1}}\frac{f(u)^2}{u}$$
$$\le\prod_p\left(1+\frac{f(p)^2}{p}\right)\sum_{n\le M}\frac{\Lambda_F(n)f(n)}{ng(n)}$$
$$\ll \mathcal{Q}_0K\log_2 M. \tag{8.3}$$

Combing (8.3) with Theorem 5, as $T\to\infty$, we have

$$S_0=\frac{T}{2\pi}d_FP_1(\log(\lambda Q^2)^{1/d_F}T)\mathcal{Q}_0(1+o(1))-\frac{T}{2\pi}\sum_{nu\le M}\frac{\Lambda_F(n)x_uy_{nu}}{nu}+o(T)$$
$$=\frac{d_FT}{2\pi}\log T\mathcal{Q}_0(1+o(1))+O\left(T\mathcal{Q}_0K\log_2 M\right).$$

Therefore,

$$\max_{\substack{F(\rho)=0\\T\le\Im\rho\le 2T}}|\zeta(\rho)|\ge\frac{|S_1|}{S_2}\gg\frac{\mathcal{Q}_1}{\mathcal{Q}_0}=\exp\left(\frac{1}{\phi(\mathrm{lcm}(q,g^*))}(1+o(1))\frac{K}{\log K^2}\right),$$

by (3) in Lemme 7.2. If $M=\exp\left(c'\sqrt{\log T}\right)$, then we can chose $K=\sqrt{\phi(\mathrm{lcm}(q,g^*))\log M\log_2 M}$, which gives

$$\frac{K}{\log K^2}\gg\sqrt{\frac{\log M}{\log_2 M}}=c''\frac{(\log T)^{1/4}}{(\log_2 T)^{1/2}},$$

with $c''=\sqrt{c'/2}$. In the second part, we can take $M=T^{\delta'}$ for some small $\delta'=\delta'(F,a)>0$ and

$$\frac{K}{\log K^2}\gg\sqrt{\frac{\log M}{\log_2 M}}\gg\sqrt{\delta'\frac{\log T}{\log_2 T}}.$$

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Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: jli135@mpim-bonn.mpg.de