THE FINAL PROBLEM:
AN IDENTITY FROM RAMANUJAN'S LOST NOTEBOOK

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ABSTRACT. In a short fragment published for the first time with his lost notebook in 1988 [19], Ramanujan offered two beautiful identities, associated, respectively, with the classical circle and divisor problems. In fact, they are analogues, with an additional variable, but not generalizations, of classical identities associated with these two famous problems. After Ramanujan's death in 1920, the lost notebook and fragments of papers of Ramanujan were sent to G.H. Hardy. We do not have any official record of what was included in this mailing, but it is likely that the aforementioned fragment was included in this parcel. If so, then it is possible that Ramanujan wrote it at the end of his life in either 1919 or 1920. On the other hand, from a paper that Hardy published in 1915 [14] on the circle problem, we are aware that by early in his stay in England, Ramanujan had a strong interest in these problems, and so the fragment may emanate from this period. Two of the present authors and S. Kim published a proof of the identity connected with the circle problem in 2013 [9]. In this paper, a proof of the second identity is given for the first time.

1. INTRODUCTION

In his retiring Presidential address to the London Mathematical Society in 1935, G.N. Watson [21], [11, pp. 325–347] provided an account of *The final problem: An account of the mock theta functions*. By "the final problem," a title borrowed from Sherlock Holmes, Watson was referring to Ramanujan's account of his "new" discovery, the mock theta functions, in his final letter to G.H. Hardy, partially reproduced in Ramanujan's *Collected Papers* [18, pp. xxxi–xxxii, 354–355] and more fully reproduced with the publication of his "lost notebook." [19, pp. 127–131].

In choosing the title of this paper, the present authors are clearly borrowing from Watson. However, for us, the "final problem" is the final entry from the "lost notebook" that remained to be proved. We are using the term "lost notebook" broadly here, for with the publication of the "lost notebook" [19] are partial unpublished manuscripts and other fragments. The entry that remained impenetrable was one of a pair of identities in an isolated fragment connected, respectively, with the *circle* and *divisor problems*. Perhaps it is best therefore to begin with very brief descriptions of these two famous problems.

Let $r_2(n)$ denote the number of representations of the positive integer n as a sum of two squares, where representations with different orders and different signs on the summands being squared are regarded as distinct. Thus, $r_2(5) = 8$, since $5 = (\pm 2)^2 + (\pm 1)^2 = (\pm 1)^2 + (\pm 2)^2$. Set $r_2(0) = 1$. Define the "error" term $P(x)$, for $x > 0$, by

$$\sum_{0 \leq n \leq x}{}' r_2(n) = \pi x + P(x),$$

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where the prime $'$ on the summation sign indicates that if x is an integer, then only $\frac{1}{2}r_2(x)$ is counted. Finding the correct order of magnitude for $P(x)$, as $x \to \infty$, is known as the *circle problem*. For a recent account of this famous unsolved problem, see the survey paper [10].

Let $d(n)$ denote the number of positive divisors of the positive integer n. For any $x > 0$, by an elementary argument counting lattice points and involving a familiar estimate for a partial sum of a harmonic series [17, p. 102, Theorem 42], we can obtain the estimate, as $x \to \infty$,

$$\sum_{n \le x}{}' d(n) = x(\log x + (2\gamma - 1)) + \frac{1}{4} + \Delta(x),$$

where $\Delta(x)$ is the "error term," where the prime $'$ on the summation sign indicates that if x is an integer, then only $\frac{1}{2}d(x)$ is counted, and where γ denotes Euler's constant. Determining the correct order of magnitude of $\Delta(x)$ as x tends to ∞ is known as the *divisor problem*. A survey of this equally famous unsolved problem can also be found in [10].

The error term $P(x)$ can be represented as an infinite series of Bessel functions, namely,

$$\sum_{0 \le n \le x}{}' r_2(n) = \pi x + \sum_{n=1}^{\infty} r_2(n) \left(\frac{x}{n}\right)^{1/2} J_1(2\pi\sqrt{nx}), \tag{1.1}$$

where the Bessel function $J_\nu(z)$ is defined by

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu + n + 1)} \left(\frac{z}{2}\right)^{\nu+2n}, \qquad 0 < |z| < \infty, \qquad \nu \in \mathbb{C}. \tag{1.2}$$

To the best of our knowledge, the identity (1.1) first appeared in Hardy's paper [14], [15, pp. 243–263], where he wrote "The form of this equation was suggested to me by Mr. S. Ramanujan, to whom I had communicated the analogous formula for $d(1) + d(2) + \cdots + d(n)$, where $d(n)$ is the number of divisors of n." Indeed, for $x > 0$, such a formula is due to G.F. Voronoï [20] and is given by

$$D(x) := \sum_{n \le x}{}' d(n) = x(\log x + 2\gamma - 1) + \frac{1}{4} + \sum_{n=1}^{\infty} d(n) \left(\frac{x}{n}\right)^{1/2} I_1(4\pi\sqrt{nx}), \tag{1.3}$$

where $I_1(z)$ is defined by

$$I_\nu(z) := -Y_\nu(z) - \frac{2}{\pi}K_\nu(z), \tag{1.4}$$

and where $Y_\nu(z)$ is the Bessel function of the second kind defined by

$$Y_\nu(z) := \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}, \quad |z| < \infty, \tag{1.5}$$

and $K_\nu(z)$ is the modified Bessel function defined by

$$K_\nu(z) := \frac{\pi}{2} \frac{e^{\pi i\nu/2}J_{-\nu}(iz) - e^{-\pi i\nu/2}J_\nu(iz)}{\sin(\nu\pi)}, \quad -\pi < \arg z < \tfrac{1}{2}\pi, \quad 0 < |z| < \infty. \tag{1.6}$$

If ν is an integer n, it is understood that we define the functions above by taking limits as $\nu \to n$ in (1.5) and (1.6).

The fragment [19, p. 335] published with the lost notebook comprises two-variable analogues of (1.1) and (1.3). In order to state these formulas, after Ramanujan, define

$$F(x) = \begin{cases} [x], & \text{if } x \text{ is not an integer,} \\ x - \frac{1}{2}, & \text{if } x \text{ is an integer.} \end{cases} \tag{1.7}$$

We first state the entry associated with the *circle problem.*

Entry 1.1. [19, p. 335] *Let $F(x)$ be defined by* (1.7), *and recall that $J_1(z)$ is defined in* (1.2). *If $0<\theta<1$ and $x>0$, then*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin(2\pi n\theta) = \pi x\left(\frac{1}{2}-\theta\right) - \frac{1}{4}\cot(\pi\theta) + \frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}. \tag{1.8}$$

To help us appreciate (1.8), by an elementary formula for $r_2(n)$ [16, p. 150],

$$\begin{aligned}\sum_{0<n\leq x}{}' r_2(n) &= 4\sum_{0<n\leq x}{}'\sum_{d|n}\sin\left(\frac{\pi d}{2}\right)\\ &= 4\sum_{0<dj\leq x}{}'\sin\left(\frac{\pi d}{2}\right)\\ &= 4\sum_{0<d\leq x}{}'\left[\frac{x}{d}\right]\sin\left(\frac{\pi d}{2}\right),\end{aligned} \tag{1.9}$$

where $[x]$ is the greatest integer less than or equal to x. Thus, the left-hand side of (1.8) can be considered as a 2-variable analogue of (1.9), while the right-hand side of (1.8) is a two-variable analogue of the series on the right side of (1.1). Entry 1.1 was first proved by the first and third authors [12] with the order of summation on the double sum *reversed* from that recorded by Ramanujan. A proof of (1.8) with the order of summation as given by Ramanujan was established seven years later by the aforementioned two authors and S. Kim [9]. They also proved a version of (1.8) with the product of the indices mn tending to ∞.

We now offer the second identity from the fragment [19, p. 335].

Entry 1.2. *Let $F(x)$ be defined by* (1.7). *Then, for $x>0$ and $0<\theta<1$,*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(2\pi n\theta) = \frac{1}{4} - x\log(2\sin(\pi\theta)) + \frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{I_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} + \frac{I_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}, \tag{1.10}$$

where $I_\nu(z)$ is defined by (1.4).

Assuming (temporarily) that x is not an integer, by an elementary argument, we see that

$$D(x) = \sum_{n\leq x}\sum_{d|n} 1 = \sum_{dj\leq x} 1 = \sum_{d\leq x}\sum_{1\leq j\leq x/d} 1 = \sum_{d\leq x}\left[\frac{x}{d}\right]. \tag{1.11}$$

Thus, the left side of (1.10) is a generalization of (1.11), while the right-hand side is a two-variable analogue of the series on the right side of (1.3).

The first and third authors along with S. Kim first proved (1.10), but with the order of summation either *reversed* and with additional conditions [8, Theorem 17], or with the product of indices mn tending to ∞ [8, Theorem 4]. Reverting to the order of summation that was recorded by Ramanujan necessitated, at least in the case of (1.8), a more sophisticated proof in [9] compared to the one in [12]. Certain difficulties, which did not arise in their proof of (1.8), arose in their attempts to prove (1.10). Also, numerical computations performed for both (1.8) and (1.10) clearly show that both sides of (1.8) match, while in the case of (1.10) all the numerical data the authors have seen so far have been inconclusive. This raises questions about whether (1.10) is actually true or not. Thus, Entry 1.2 has remained unproved and has had the distinction in recent times for being the only unproved claim of Ramanujan from [19].

The purpose of this paper is to prove Entry 1.2, i.e., to solve the *final problem* in the lost notebook.

Theorem 1.3. *Ramanujan's Entry 1.2 is true.*

2. Sketch of Proof

If we employ the asymptotic formulas (4.4) and (4.5) for the Bessel functions on the right-hand side of (1.10), we see that the series does not converge absolutely. We therefore consider a more general series than that on the right-hand side of (1.10) in which the powers of m and $(n+\theta)$ (or $(n+1-\theta)$) are replaced by general complex variables s and w, respectively. We obtain analytic continuations of this more general function (4.1). In particular, we show that (4.1) converges uniformly with respect to θ in any compact subinterval of $(0,1)$ provided that $\mathrm{Re}(s) > \frac{1}{4}$, $\mathrm{Re}(w) > \frac{1}{4}$, and $\mathrm{Re}(s)+\mathrm{Re}(w) > \frac{25}{26}$ if x is an integer, and $\mathrm{Re}(s) > \frac{1}{4}$, $\mathrm{Re}(w) > \frac{1}{4}$, and $\mathrm{Re}(s)+\mathrm{Re}(w) > \frac{5}{6}$ if x is not an integer. Both the left and right sides of (1.10) are thus continuous functions of θ on $(0,1)$.

It is helpful to work with continuous functions on $[0,1]$, instead of functions continuous only on $(0,1)$. We therefore first multiply both sides of the proposed identity (1.10) by $\sin^2(\pi\theta)$ in order to extend the domain of continuity $0<\theta<1$ to $0\leq\theta\leq 1$. We then rewrite the amended proposed identity by isolating the series of Bessel functions on one side of the equation. Next, we calculate the Fourier series of each side of the new amended proposed identity and show that they are equal. Because each side of the proposed identity is continuous, we can thus conclude from the theory of Fourier series that the two sides are identical for $0<\theta<1$.

In the next section we shall assume the necessary continuity of all functions involved and calculate the Fourier series for each side of the slightly amended proposed identity and show that they are equal. Demonstrating the uniform convergence of the extended double series of Bessel functions is reserved for the last section of our paper.

3. Calculation of Fourier Series

Define, for $0<\theta<1$ and $x>0$,

$$G_1(\theta) := \sin^2(\pi\theta)\left(\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(2\pi n\theta) - \frac{1}{4} + x\log(2\sin(\pi\theta))\right). \tag{3.1}$$

Since $\lim_{\theta\to 0}\sin^2\pi\theta\log(2\sin\pi\theta)=0$, we see that $G_1(\theta)$ is a continuous function of θ in $[0,1]$ satisfying $G_1(\theta)=G_1(1-\theta)$. In order to make our next definition, we need to first consult Theorem 4.1 in which we set $w=\frac{1}{2}$. thus, if x is an integer, assume that $\mathrm{Re}(s)>\frac{6}{13}$, while if x is not an integer, assume that $\mathrm{Re}(s)>\frac{1}{3}$. We make these assumptions throughout Section 3. Therefore, define, for $0<\theta<1$ and $x>0$,

$$G(x,\theta,s) := \sum_{m=1}^{\infty} a(x,\theta,m)m^{-s}, \tag{3.2}$$

where

$$a(x,\theta,m)=\sum_{n=0}^{\infty}\left\{\frac{I_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{n+\theta}}+\frac{I_1(4\pi\sqrt{m(n+1-\theta)x})}{\sqrt{n+1-\theta}}\right\}.$$

We now calculate the Fourier series of $G_1(\theta)$ and $G(x,\theta,\frac{1}{2})$ and show that they are identical.

Theorem 3.1. *For $0<\theta<1$ and $x>0$, we have*

$$G_1(\theta)=\sum_{n=0}^{\infty}c_n\cos(2\pi n\theta), \tag{3.3}$$

where

$$c_0=-\frac{1}{8}+\frac{x}{4}-\frac{1}{4}F(x), \tag{3.4}$$

$$c_1=\frac{1}{2}F(x)-\frac{1}{4}F\left(\frac{x}{2}\right)+\frac{1}{8}-\frac{3x}{8}, \tag{3.5}$$

and, for $n\geq 2$,

$$c_n=\frac{1}{2}F\left(\frac{x}{n}\right)-\frac{1}{4}F\left(\frac{x}{n+1}\right)-\frac{1}{4}F\left(\frac{x}{n-1}\right)+\frac{x}{4}\left(\frac{1}{n+1}+\frac{1}{n-1}-\frac{2}{n}\right), \tag{3.6}$$

where $F(x)$ is defined in (1.7).

Proof. By [23, p. 190, Ex. 6] or [7, Lemma 3.7], for $0<\theta<1$,

$$\sum_{k=1}^{\infty}\frac{\cos(2\pi k\theta)}{k}=-\log(2\sin(\pi\theta)). \tag{3.7}$$

Furthermore, for each nonnegative integer k,

$$\sin^2\pi\theta\cos(2\pi k\theta)=\frac{1}{2}\cos(2\pi k\theta)-\frac{1}{4}\cos(2\pi(k-1)\theta)-\frac{1}{4}\cos(2\pi(k+1)\theta). \tag{3.8}$$

It follows that for $0<\theta<1$,

$$\begin{aligned}\sin^2(\pi\theta)\log(2\sin(\pi\theta))&=-\sum_{k=1}^{\infty}\frac{\cos(2\pi k\theta)\sin^2(\pi\theta)}{k}\\&=\sum_{k=1}^{\infty}\frac{\cos(2\pi(k+1)\theta)+\cos(2\pi(k-1)\theta)-2\cos(2\pi k\theta)}{4k}\\&=\sum_{k=2}^{\infty}\frac{1}{4}\cos(2\pi k\theta)\left(\frac{1}{k+1}+\frac{1}{k-1}-\frac{2}{k}\right)+\frac{1}{4}-\frac{3\cos(2\pi\theta)}{8}.\end{aligned} \tag{3.9}$$

Thus, by (3.1) and (3.9),

$$\begin{aligned}G_1(\theta)=&\sum_{n=1}^{\infty}F\left(\frac{x}{n}\right)\left(\frac{1}{2}\cos(2\pi n\theta)-\frac{1}{4}\cos(2\pi(n-1)\theta)-\frac{1}{4}\cos(2\pi(n+1)\theta)\right)\\&-\frac{1}{8}(1-\cos(2\pi\theta))+x\sum_{k=2}^{\infty}\frac{1}{4}\cos(2\pi k\theta)\left(\frac{1}{k+1}+\frac{1}{k-1}-\frac{2}{k}\right)+\frac{x}{4}-\frac{3x\cos(2\pi\theta)}{8}.\end{aligned} \tag{3.10}$$

If we now calculate the Fourier coefficients c_n in (3.10), $n\geq 0$, we readily deduce (3.4)–(3.6). □

Since $G(x,\theta,s)$ is analytic in $\mathrm{Re}(s) > \frac{6}{13}$, it follows that $G(x,\theta,\frac{1}{2})$ is well defined and satisfies $G(x,\theta,\frac{1}{2}) = G(x,1-\theta,\frac{1}{2})$. Define

$$\widetilde{G_1}(\theta) := G(x,\theta,\tfrac{1}{2}) = \frac{\sqrt{x}}{2}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{I_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}} + \frac{I_1(4\pi\sqrt{m(n+1-\theta)x})}{\sqrt{m(n+1-\theta)}}\right\}\sin^2(\pi\theta). \tag{3.11}$$

Theorem 3.2. *For $0 < \theta < 1$ and $x > 0$, define*

$$\widetilde{G_1}(\theta) = \frac{1}{2}b_0 + \sum_{j=1}^{\infty} b_j \cos(2\pi j\theta). \tag{3.12}$$

Then

$$\frac{1}{2}b_0 = -\frac{1}{8} + \frac{x}{4} - \frac{1}{4}F(x), \tag{3.13}$$

$$b_1 = \frac{1}{2}F(x) - \frac{1}{4}F\left(\frac{x}{2}\right) + \frac{1}{8} - \frac{3x}{8}, \tag{3.14}$$

and, for $j \geq 2$,

$$b_j = \frac{1}{2}F\left(\frac{x}{j}\right) - \frac{1}{4}F\left(\frac{x}{j+1}\right) - \frac{1}{4}F\left(\frac{x}{j-1}\right) + \frac{x}{4}\left(\frac{1}{j+1} + \frac{1}{j-1} - \frac{2}{j}\right), \tag{3.15}$$

where $F(x)$ is defined in (1.7).

In order to calculate these Fourier coefficients, we need the following three lemmas.

Lemma 3.3. [7, Lemma 3.4] *Let I_1 be defined in* (1.4), *and for $b, c > 0$, set*

$$A = \frac{c^2}{8b}.$$

Then

$$\int_0^{\infty} \cos(bx^2) I_1(cx)dx = \frac{1}{4}\sqrt{\frac{2}{bA}}\sin(2A).$$

Lemma 3.4. [7, Lemma 3.5] *Let I_1 be defined in* (1.4). *Then*

$$\int_0^{\infty} I_1(u)du = 0.$$

Lemma 3.5. *For any real number y,*

$$-\sum_{m=1}^{\infty}\frac{\sin(2\pi my)}{\pi m} = \begin{cases} 0, & \text{if } y \text{ is an integer,} \\ y - [y] - \frac{1}{2}, & \text{if } y \text{ is not an integer.} \end{cases}$$
$$= -F(x) + x - \frac{1}{2}, \tag{3.16}$$

where $F(x)$ is defined in (1.7).

Proof. From the definitions (3.11) and (3.12), the Fourier coefficients $b_j, j \geq 0$, are given by

$$\begin{aligned} b_j &= 2\sqrt{x}\int_0^{1/2}\sum_{m=1}^\infty\sum_{n=0}^\infty\left\{\frac{I_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}}+\frac{I_1(4\pi\sqrt{m(n+1-\theta)x})}{\sqrt{m(n+1-\theta)}}\right\}\sin^2(\pi\theta)\cos(2\pi j\theta)d\theta \\ &= \sqrt{x}\sum_{m=1}^\infty\sum_{n=0}^\infty\int_0^{1/2}\left\{\frac{I_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}}+\frac{I_1(4\pi\sqrt{m(n+1-\theta)x})}{\sqrt{m(n+1-\theta)}}\right\} \\ &\quad\times(\cos(2\pi j\theta)-\tfrac{1}{2}\cos(2\pi\theta(j+1)-\tfrac{1}{2}\cos(2\pi\theta(j-1))d\theta \\ &=: S_1+S_2+S_3. \end{aligned} \tag{3.17}$$

We focus on S_1. In the first set of integrals, set $u = 4\pi\sqrt{m(n+\theta)x}$, which gives

$$\frac{d\theta}{\sqrt{m(n+\theta)}} = \frac{du}{2\pi m\sqrt{x}},$$

and in the second set of integrals, set $u = 4\pi\sqrt{m(n+1-\theta)x}$, which gives

$$\frac{d\theta}{\sqrt{m(n+1-\theta)}} = -\frac{du}{2\pi m\sqrt{x}}.$$

Hence,

$$\begin{aligned} S_1 &= \sqrt{x}\sum_{m=1}^\infty\sum_{n=0}^\infty\int_0^{1/2}\left\{\frac{I_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}}+\frac{I_1(4\pi\sqrt{m(n+1-\theta)x})}{\sqrt{m(n+1-\theta)}}\right\}\cos(2\pi j\theta)d\theta \\ &= \sum_{m=1}^\infty\sum_{n=0}^\infty\frac{1}{2\pi m}\int_{4\pi\sqrt{mnx}}^{4\pi\sqrt{m(n+1/2)x}} I_1(u)\cos\left(2\pi j\left(\frac{u^2}{16\pi^2 mx}-n\right)\right)du \\ &\quad - \int_{4\pi\sqrt{m(n+1)x}}^{4\pi\sqrt{m(n+1/2)x}} I_1(u)\cos\left(2\pi j\left(n+1-\frac{u^2}{16\pi^2 mx}\right)\right)du \\ &= \sum_{m=1}^\infty\sum_{n=0}^\infty\frac{1}{2\pi m}\int_{4\pi\sqrt{mnx}}^{4\pi\sqrt{m(n+1)x}} I_1(u)\cos\left(\frac{u^2 j}{8\pi mx}\right)du \\ &= \sum_{m=1}^\infty\frac{1}{2\pi m}\int_0^\infty I_1(u)\cos\left(\frac{u^2 j}{8\pi mx}\right)du. \end{aligned} \tag{3.18}$$

Upon an application of Lemma 3.3 with $c = 1$ and $b = j/(8\pi mx)$, if $j > 0$, and Lemma 3.4 for $j = 0$, we find that (3.18) can be reduced to

$$S_1 = \begin{cases} \displaystyle\sum_{m=1}^\infty \frac{1}{2\pi m}\sin\left(\frac{2\pi mx}{j}\right), & \text{if } j > 0, \\ 0, & \text{if } j = 0. \end{cases} \tag{3.19}$$

Similar calculations may be effected for S_2 and S_3. Thus, by (3.19), for $j \geq 0$,

$$S_2 = -\frac{1}{2}\sum_{m=1}^\infty\frac{1}{2\pi m}\sin\left(\frac{2\pi mx}{j+1}\right). \tag{3.20}$$

By (3.19),

$$S_3 = \begin{cases} -\dfrac{1}{2}\displaystyle\sum_{m=1}^{\infty} \frac{1}{2\pi m} \sin\left(\frac{2\pi m x}{|j-1|}\right), & \text{if } j \neq 1, \\ 0, & \text{if } j = 1. \end{cases} \tag{3.21}$$

We now return to (3.17) and employ (3.19)–(3.21) to evaluate b_j. Thus,

$$b_0 = \sum_{m=1}^{\infty} \frac{1}{2\pi m} \left\{-\sin(2\pi m x)\right\},$$
$$b_1 = \sum_{m=1}^{\infty} \frac{1}{2\pi m} \left\{\sin(2\pi m x) - \frac{1}{2}\sin(\pi m x)\right\},$$
$$b_j = \sum_{m=1}^{\infty} \frac{1}{2\pi m} \left\{\sin\left(\frac{2\pi m x}{j}\right) - \frac{1}{2}\sin\left(\frac{2\pi m x}{j+1}\right) - \frac{1}{2}\sin\left(\frac{2\pi m x}{j-1}\right)\right\}, \text{ if } j \geq 2.$$

Using Lemma 3.5, we now express each of the evaluations above in terms of $F(x)$, and so we find that

$$b_0 = -\frac{1}{2}F(x) + \frac{x}{2} - \frac{1}{4},$$
$$b_1 = \frac{1}{8} - \frac{3x}{8} + \frac{1}{2}F(x) - \frac{1}{4}F\left(\frac{x}{2}\right),$$
$$b_j = \frac{1}{2}F\left(\frac{x}{j}\right) - \frac{1}{4}F\left(\frac{x}{j+1}\right) - \frac{1}{4}F\left(\frac{x}{j-1}\right) + \frac{x}{4}\left(\frac{1}{j+1} + \frac{1}{j-1} - \frac{2}{j}\right), \quad j \geq 2. \tag{3.22}$$

This concludes the proof of Theorem 3.2. □

Comparing the Fourier coefficients of $G_1(\theta)$ and $\widetilde{G_1}(\theta)$ in Theorems 3.1 and 3.2, respectively, we see that they are identical. Since $G_1(\theta)$ and $\widetilde{G_1}(\theta)$ are continuous functions, we conclude from Parseval's Theorem [23, p. 182] that $G_1(\theta) = \widetilde{G_1}(\theta)$ for all $0 < \theta < 1$, which completes the proof of Entry 1.2.

4. A Two Variable Generalization and Its Uniform Convergence

Define

$$G(x,\theta,s,w) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left(\frac{a_1(x,\theta,m,n)}{m^s(n+\theta)^w} + \frac{a_2(x,\theta,m,n)}{m^s(n+1-\theta)^w}\right), \tag{4.1}$$

where

$$a_1(x,\theta,m,n) = I_1(4\pi\sqrt{m(n+\theta)x}), \tag{4.2}$$
$$a_2(x,\theta,m,n) = I_1(4\pi\sqrt{m(n+1-\theta)x}), \tag{4.3}$$

where $I_1(z)$ is defined by (1.4). We want to determine the values of (s,w) for which $G(x,\theta,s,w)$ converges.

Theorem 4.1. *Let $G(x,\theta,s,w)$ be defined above. Assume that $\mathrm{Re}(s) > \frac{1}{4}$ and $\mathrm{Re}(w) > \frac{1}{4}$. Furthermore, if x is an integer, assume that $\mathrm{Re}(s)+\mathrm{Re}(w) > \frac{25}{26}$, while if x is not an integer, assume that $\mathrm{Re}(s)+\mathrm{Re}(w) > \frac{5}{6}$. Then the series converges uniformly with respect to θ in any compact subinterval of $(0,1)$.*

Proof. We prove Theorem 4.1 in several steps.

As $|z| \to \infty$ [22, p. 199, p. 202],

$$Y_1(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sin\left(z - \frac{3}{4}\pi\right) + O\left(\frac{1}{z^{3/2}}\right), \tag{4.4}$$

$$K_1(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left(1 + \frac{3}{8z} + O\left(\frac{1}{z^2}\right)\right), \quad |\arg z| < 3\pi/2. \tag{4.5}$$

Therefore, from (1.4), as z approaches infinity,

$$I_1(z) = -\left(\frac{2}{\pi z}\right)^{1/2} \sin\left(z - \frac{3}{4}\pi\right) + O\left(\frac{1}{z^{3/2}}\right). \tag{4.6}$$

If we employ (4.6) in (4.1), we see that the study of the convergence of $G(x, \theta, s, w)$ reduces to the study of

$$S_1(a, \theta, s, w) := \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \frac{1}{m^{s+1/4}} \left(\frac{\sin(a\sqrt{m(n+\theta)} - \frac{3}{4}\pi)}{(n+\theta)^{w+1/4}} + \frac{\sin(a\sqrt{m(n+1-\theta)} - \frac{3}{4}\pi)}{(n+1-\theta)^{w+1/4}} \right), \tag{4.7}$$

where $x > 0$ and $a = 4\pi\sqrt{x}$. The proof of Entry 1.2 then rests upon the study of the uniform convergence of the double series $S_1(a, \theta, s, w)$ and its analytic continuation with respect to s and w.

4.1. **Large Values of n.** We first examine (4.7) for large values of n. We apply the Euler–Maclaurin summation formula [1, p. 619]. Let M and N be integers with $M < N$ and suppose that $0 < \theta < 1$. Then

$$\sum_{n=M+1}^{N} \frac{\sin(a\sqrt{m(n+\theta)} - \frac{3}{4}\pi)}{(n+\theta)^{w+1/4}} = \int_{M+\theta}^{N+\theta} \frac{\sin(a\sqrt{mt} - \frac{3}{4}\pi)}{t^{w+1/4}} dt + \int_{M+\theta}^{N+\theta} \{t - \theta\} \frac{d}{dt}\left(\frac{\sin(a\sqrt{mt} - \frac{3}{4}\pi)}{t^{w+1/4}}\right) dt. \tag{4.8}$$

From the derivative,

$$\frac{d}{dt}\left(\frac{\sin(a\sqrt{mt} - \frac{3}{4}\pi)}{t^{w+1/4}}\right) = \frac{(4w+1)\sin\left(a\sqrt{mt} + \frac{\pi}{4}\right) - 2a\sqrt{mt}\cos\left(a\sqrt{mt} + \frac{\pi}{4}\right)}{4t^{w+5/4}} = O\left(\frac{a\sqrt{m}}{t^{\mathrm{Re}(w)+3/4}}\right),$$

we readily find that

$$\int_{M+\theta}^{N+\theta} \{t - \theta\} \frac{d}{dt}\left(\frac{\sin(a\sqrt{mt} - \frac{3}{4}\pi)}{t^{w+1/4}}\right) dt = O\left(\frac{a\sqrt{m}}{(M+\theta)^{\mathrm{Re}(w)-1/4}}\right). \tag{4.9}$$

Let $u = a\sqrt{mt}$; then $t = \frac{u^2}{a^2m}$ and $dt = \frac{2u}{a^2m}du$. Thus,

$$\int_{M+\theta}^{N+\theta} \frac{\sin(a\sqrt{mt} - \frac{3}{4}\pi)}{t^{w+1/4}}dt = 2(a^2m)^{w-3/4} \int_{a\sqrt{m(M+\theta)}}^{a\sqrt{m(N+\theta)}} \frac{\sin(u - \frac{3}{4}\pi)}{u^{2w-1/2}}du$$
$$= -\sqrt{2}(a^2m)^{w-3/4} \int_{a\sqrt{m(M+\theta)}}^{a\sqrt{m(N+\theta)}} \frac{\sin u + \cos u}{u^{2w-1/2}}du. \tag{4.10}$$

From the fact that

$$\int_A^B \frac{\sin u + \cos u}{u^{2w-1/2}}du$$
$$= \frac{\cos A - \sin A}{A^{2w-1/2}} - \frac{\cos B - \sin B}{B^{2w-1/2}} - (2w - 1/2) \int_A^B \frac{\cos u - \sin u}{u^{2w+1/2}}du$$
$$= O_w\left(\frac{1}{A^{2\operatorname{Re}(w)-1/2}} + \frac{1}{B^{2\operatorname{Re}(w)-1/2}}\right),$$

as $A, B \to \infty$, we see from (4.10) that

$$\int_{M+\theta}^{N+\theta} \frac{\sin(a\sqrt{mt} - \frac{3}{4}\pi)}{t^{w+1/4}}dt = O\left((a^2m)^{-1/2}\left(\frac{1}{(M+\theta)^{2\operatorname{Re}(w)-1/2}} + \frac{1}{(N+\theta)^{2\operatorname{Re}(w)-1/2}}\right)\right). \tag{4.11}$$

Employing (4.9) and (4.11) in (4.8), we deduce that, as $M \to \infty$,

$$\sum_{n=M}^{\infty} \frac{\sin(a\sqrt{m(n+\theta)} - \frac{3}{4}\pi)}{(n+\theta)^{w+1/4}} = \lim_{N\to\infty} \sum_{n=M}^{N} \frac{\sin(a\sqrt{m(n+\theta)} - \frac{3}{4}\pi)}{(n+\theta)^{w+1/4}} = O\left(\frac{a\sqrt{m}}{(M+\theta)^{\operatorname{Re}(w)-1/4}}\right), \tag{4.12}$$

Of course, an analogous result holds when θ is replaced by $1-\theta$. If we set $M = [m^{1/(\operatorname{Re}(w)-1/4)}]$ in (4.12) and its companion with θ replaced by $1-\theta$, we can then conclude that

$$\sum_{n\geq m^{1/(\operatorname{Re}(w)-1/4)}} \frac{1}{m^{s+1/4}}\left(\frac{\sin(a\sqrt{m(n+\theta)} - \frac{3}{4}\pi)}{(n+\theta)^{w+1/4}} + \frac{\sin(a\sqrt{m(n+1-\theta)} - \frac{3}{4}\pi)}{(n+1-\theta)^{w+1/4}}\right)$$
$$= O\left(\frac{a}{m^{\operatorname{Re}(s)+3/4}}\right),$$

as $m \to \infty$. Therefore, in our study of the uniform convergence of $S_1(x, \theta, s, w)$, we may replace this sum in our investigation by the sum

$$S_2(a, \theta, s, w) := \sum_{m=1}^{\infty} \sum_{0\leq n\leq m^{1/(\operatorname{Re}(w)-1/4)}} \frac{1}{m^{s+1/4}}\left(\frac{\sin(a\sqrt{m(n+\theta)} - \frac{3}{4}\pi)}{(n+\theta)^{w+1/4}}\right.$$
$$\left. + \frac{\sin(a\sqrt{m(n+1-\theta)} - \frac{3}{4}\pi)}{(n+1-\theta)^{w+1/4}}\right). \tag{4.13}$$

4.2. **Small Values of n.** Next, we show that we can reduce our study of (4.13) to the examination of

$$S_3(a,\theta,s,w) := \sum_{m=1}^{\infty} \sum_{m^{1-\delta}\le n\le m^{1+\delta}} \left(\frac{\sin\left(a\sqrt{m(n+\theta)}-\frac{3}{4}\pi\right)}{m^{s+1/4}(n+\theta)^{w+1/4}} + \frac{\sin\left(a\sqrt{m(n+1-\theta)}-\frac{3}{4}\pi\right)}{m^{s+1/4}(n+1-\theta)^{w+1/4}} \right), \tag{4.14}$$

where $0<\delta<1$ will be more precisely defined later. In [9, Equation (4.1)], the following type of sum was considered:

$$S(\alpha,\beta,\mu,H_1,H_2) := \sum_{H_1<m\le H_2} \frac{\cos(\alpha\sqrt{(m(n+\mu))}+\beta)}{(m+\mu)^{s+1/4}}, \tag{4.15}$$

where $\alpha>0$, $\beta\in\mathbb{R}$, $\mu\in[0,1]$, and $H_1<H_2$ are large positive integers. It was shown in [9] that, subject to

$$c_1 \le \alpha \le c_2 H_1^{(1-\delta)/2}, \tag{4.16}$$

where $c_1>0, c_2>0$ are constants and δ is a fixed small positive real number, that the sum in (4.15) satisfies [9, Equation (4.12)]

$$S(\alpha,\beta,\mu,H_1,H_2) = O\left(\frac{1}{\alpha H_1^{\mathrm{Re}(s)-1/4}}\right). \tag{4.17}$$

Consider

$$S_4(a,\theta,\delta,s,w) := \sum_{m=1}^{\infty} \sum_{0\le n<m^{1-\delta}} \frac{1}{m^{s+1/4}} \left(\frac{\sin(a\sqrt{m(n+\theta)}-\frac{3}{4}\pi)}{(n+\theta)^{w+1/4}} + \frac{\sin(a\sqrt{m(n+1-\theta)}-\frac{3}{4}\pi)}{(n+1-\theta)^{w+1/4}} \right). \tag{4.18}$$

At this point, we follow the argument from [9, Equations (4.14)–(4.16)]. Thus, as in [9], we apply Cauchy's criterion, and we take advantage of two facts. One is that the two sin-functions that appear in the numerators on the right-hand side of (4.15) can be written in terms of cos-functions via the identity $\sin t = \cos(t-\frac{1}{2}\pi)$, so that the bound (4.17) is directly applicable in our present context, but with a different choice of β. The second fact concerns the very convenient way the analogue of [9, Equation (4.15)] in our present situation depends on w. More precisely,

$$\frac{1}{(n+\theta)^{w+1/4}} \qquad \text{and} \qquad \frac{1}{(n+1-\theta)^{w+1/4}},$$

respectively, from (4.18), appear as common factors in front of the inner sums on the right side of the analogue of [9, Equation (4.15)]. Lastly, the exterior sums over n, bounded in [9, Equation (4.16)] by

$$O_{a,\delta}\left(\sum_{1\le n\le M_1^{1-\delta}} \frac{1}{n^{5/4}M_1^{\mathrm{Re}\,s-1/4}}\right) + O_{a,\delta}\left(\sum_{M_1^{1-\delta}<n\le M_2^{1-\delta}} \frac{1}{n^{5/4+(\mathrm{Re}\,s-1/4)/(1-\delta)}}\right)$$

will be replaced in our present case by

$$O_{a,\delta}\left(\sum_{1\le n\le M_1^{1-\delta}}\frac{1}{n^{w+3/4}M_1^{\operatorname{Re}s-1/4}}\right)+O_{a,\delta}\left(\sum_{M_1^{1-\delta}<n\le M_2^{1-\delta}}\frac{1}{n^{w+3/4+(\operatorname{Re}s-1/4)/(1-\delta)}}\right).$$

Therefore, the argument leading from Equation (4.16) to Equation (4.17) in [9] is valid in our present case as well, for any pair (s,w) of complex numbers for which $\operatorname{Re}s>\frac14$ and $\operatorname{Re}w>\frac14$.

Next, as explained at the beginning of [9, Section 5], the range

$$m^{1+\delta}<n<m^{11/3}\log^5 m$$

can be handled in the same way, by employing again the Euler–Maclaurin summation formula of sufficiently high order, and with the roles of m and n reversed. The same argument applies in our present case, allowing us to successfully handle the range

$$m^{1+\delta}<n<m^{1/(\operatorname{Re}w-1/4)}.$$

In conclusion, it remains to consider the range

$$m^{1-\delta}<n<m^{1+\delta},$$

that is, to consider the sum (4.14).

4.3. **Further Reductions.** Define

$$S_6(a,\theta,\delta,s,w):=\sum_{m=1}^{\infty}\sum_{m^{1-\delta}<n\le m^{1+\delta}}\frac{\sin\left(a\sqrt{m(n+\frac12)}-\frac34\pi\right)\cos\left(\frac{a(1-2\theta)}{4}\sqrt{\frac{m}{n}}\right)}{m^{s+1/4}n^{w+1/4}}. \tag{4.19}$$

By employing the same analysis as we did in Section 5 of [9], we can show that we may replace our study of the uniform convergence of $S_3(a,\theta,s,w)$ to that of $S_6(a,\theta,\delta,s,w)$ when θ belongs to an arbitrary compact subinterval in $(0,1)$, and s and w are fixed with $\operatorname{Re}s>\frac14$ and $\operatorname{Re}w>\frac14$. In order to see this, notice that each of the two fractions on the right side of (4.14) is bounded in absolute value by $1/m^{\operatorname{Re}s+1/4}n^{\operatorname{Re}w+1/4}$. The error made when one approximates each of these two fractions by the corresponding fraction in (4.19) is roughly n times smaller, so these error terms are bounded by $1/m^{\operatorname{Re}s+1/4}n^{\operatorname{Re}w+5/4}$. Next, if we sum these error terms over n, with n going from $m^{1-\delta}$ to $m^{1+\delta}$, their sum is bounded by $1/m^{\operatorname{Re}s+1/4}m^{(1-\delta)(\operatorname{Re}w+1/4)}$. Lastly, summing these bounds over all m, we find that their total sum is convergent provided $\operatorname{Re}s>1/4$, $\operatorname{Re}w>1/4$, and δ is taken to be small enough (depending on $\operatorname{Re}s$ and $\operatorname{Re}w$). Thus the reduction from (4.14) to (4.19) is legitimate, for such choices of s,w, and δ.

Next, we replace $S_6(a,\theta,\delta,s,w)$ by another series, where the double sum is performed over a union of rectangles. Let

$$b=\frac{a(1-2\theta)}{4}$$

and define

$$S_7(a,\theta,\delta,s,w):=\sum_{r=1}^{\infty}\sum_{2^r\le m\le 2^{r+1}}\sum_{2^{r(1-\delta)}\le n\le 2^{(r+1)(1+\delta)}}\frac{\cos\left(b\sqrt{\frac{m}{n}}\right)\sin\left(a\sqrt{m(n+\frac12)}-\frac34\pi\right)}{m^{s+1/4}m^{w+1/4}}. \tag{4.20}$$

The extra terms with $2^{r(1-\delta)}\le n<m^{1-\delta}$ and $m^{1+\delta}\le n\le 2^{(r+1)(1+\delta)}$ will not influence our study of the uniform convergence of the series.

4.4. **Breaking the Range of Summation.** Fix a large real number R, and restrict (s,w) to be inside the circle $|s|^2+|w|^2<R$. Denote by $S_{7,M}(a,\theta,\delta,s,w)$ the partial sum in (4.20) where m is restricted to $1\le m\le M$. Let $M_1<M_2$ be large, and denote $r_1=\left\lfloor\frac{\log M_1}{\log 2}\right\rfloor$ and $r_2=\left\lfloor\frac{\log M_2}{\log 2}\right\rfloor$. Then $S_{7,M_2}-S_{7,M_1}$ can be written as a sum over integral pairs (m,n) in the union of r_2-r_1+1 rectangles. Let

$$\begin{aligned}R_0&=(M_1,2^{r_1+1})\times[2^{(1-\delta)r_1},2^{(r_1+1)(1+\delta)}],\\ R_j&=[2^{r_1+j},2^{r_1+j+1})\times[2^{(r_1+j)(1-\delta)},2^{(r_1+j+1)(1+\delta)}],\quad 1\le j<r_2-r_1,\end{aligned}$$

and

$$R_{r_2-r_1}=[2^{r_2},M_2]\times[2^{r_2(1-\delta)},2^{r_2(1+\delta)}].$$

Then

$$\begin{aligned}&S_{7,M_2}(a,\theta,\delta,s,w)-S_{7,M_1}(a,\theta,\delta,s,w)\\ &=\sum_{j=0}^{r_2-r_1}\sum_{(m.n)\in R_j}\frac{\cos\left(b\sqrt{\frac{m}{n}}\right)\sin\left(a\sqrt{m(n+\frac12)}-\frac34\pi\right)}{m^{s+1/4}n^{w+1/4}}.\end{aligned}\tag{4.21}$$

Without loss of generality, we can fix an R_j for some $1\le j\le r_2-r_1-1$. The cases where $j=0$ and $j=r_2-r_1$ can be examined in a similar fashion. Denote $T=2^{r_1+j}$ for simplicity. Then the inner sum in (4.21), which we denote by $\Sigma_{a,b,\delta,s,w,T}$, has the form

$$\Sigma_{a,b,\delta,s,w,T}=\sum_{T\le m<2T}\sum_{T^{1-\delta}\le n\le(2T)^{1+\delta}}\frac{\cos\left(b\sqrt{\frac{m}{n}}\right)\sin\left(a\sqrt{m(n+\frac12)}-\frac34\pi\right)}{m^{s+1/4}n^{w+1/4}}.\tag{4.22}$$

Fix a number λ, $0<\lambda<\frac12$, whose precise value will be given later. Denote, $L=\lfloor T^\lambda\rfloor$. Then we subdivide the rectangle $[T,2T)\times[T^{1-\delta},(2T)^{1+\delta}]$ into squares of size $L\times L$. Let

$$\begin{aligned}T_1&:=\left\lfloor\frac{T}{L}\right\rfloor+1,\qquad T_2:=\left\lfloor\frac{2T}{L}\right\rfloor-1,\\ T_3&:=\left\lfloor\frac{T^{1-\delta}}{L}\right\rfloor+1,\quad T_4:=\left\lfloor\frac{(2T)^{1+\delta}}{L}\right\rfloor-1.\end{aligned}\tag{4.23}$$

For each $m_1\in\{T_1,T_1+1,\dots,T_2\}$ and $n_1\in\{T_3,T_3+1,\dots,T_4\}$, denote

$$\Sigma_{m_1,n_1}:=\sum_{Lm_1\le m<L(m_1+1)}\sum_{Ln_1\le n<L(n_1+1)}\frac{\cos\left(b\sqrt{\frac{m}{n}}\right)\sin\left(a\sqrt{m(n+\frac12)}-\frac34\pi\right)}{m^{s+1/4}n^{w+1/4}}.\tag{4.24}$$

Note that each integral pair of points $(m,n)\in[T,2T)\times[T^{1-\delta},(2T)^{1+\delta}]$ that does not belong to any of the squares $[Lm_1,L(m_1+1))\times[Ln_1,L(n_1+1))$ with $T_1\le m_1\le T_2$ and $T_3\le n_1\le T_4$ is at most a distance L from the four sides of the rectangle $[T,2T)\times[T^{1-\delta},(2T)^{1+\delta}]$. Therefore, for $\mathrm{Re}(w)>\frac14$,

$$\left|\Sigma_{a,b,\delta,s,w,T}-\sum_{T_1\le m\le T_2}\sum_{T_3\le n\le T_4}\Sigma_{m_1,n_1}\right|$$

$$= O\left(\sum_{\substack{|m-T|\le L \\ \text{or } |m-2T|\le L}} \sum_{T^{1-\delta}\le n\le (2T)^{1+\delta}} \frac{1}{m^{\mathrm{Re}(s)+1/4} n^{\mathrm{Re}(w)+1/4}}\right)$$

$$+ O\left(\sum_{\substack{|n-T^{1-\delta}|\le L \\ \text{or } |n-(2T)^{1+\delta}|\le L}} \sum_{T\le m\le 2T} \frac{1}{m^{\mathrm{Re}(s)+1/4} n^{\mathrm{Re}(w)+1/4}}\right)$$

$$= O\left(\frac{L}{T^{\mathrm{Re}(s)+1/4}} T^{(1+\delta)(3/4-\mathrm{Re}(w))}\right) + O\left(\frac{L}{T^{(1-\delta)(\mathrm{Re}(w)+1/4)}} T^{3/4-\mathrm{Re}(s)}\right)$$

$$= O\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-\lambda-(3/4-\mathrm{Re}(w))\delta}}\right) + O\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-\lambda-(1/4+\mathrm{Re}(w))\delta}}\right)$$

$$= O\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-\lambda-(1/4+\mathrm{Re}(w))\delta}}\right). \tag{4.25}$$

We first fix λ and then fix δ depending on λ. Fix $m_1 \in \{T_1, T_1+1, \ldots, T_2\}$ and $n_1 \in \{T_3, T_3+1, \ldots, T_4\}$. For each m and n, with $Lm_1 \le m \le L(m_1+1)$ and $Ln_1 \le n \le L(n_1+1)$, as $T \to \infty$,

$$\frac{1}{m^{\mathrm{Re}(s)+1/4}} = \frac{1}{L^{\mathrm{Re}(s)+1/4} m_1^{\mathrm{Re}(s)+1/4}}\left(1 + O_R\left(\frac{1}{T^{1-\lambda}}\right)\right), \tag{4.26}$$

$$\frac{1}{n^{\mathrm{Re}(w)+1/4}} = \frac{1}{L^{\mathrm{Re}(w)+1/4} n_1^{\mathrm{Re}(w)+1/4}}\left(1 + O_R\left(\frac{1}{T^{1-\lambda-\delta}}\right)\right), \tag{4.27}$$

$$\sqrt{\frac{m}{n}} = \sqrt{\frac{m_1}{n_1}}\left(1 + O_R\left(\frac{1}{T^{1-\lambda-\delta}}\right)\right), \tag{4.28}$$

$$\cos\left(b\sqrt{\frac{m}{n}}\right) = \cos\left(b\sqrt{\frac{m_1}{n_1}}\right) + O_{x,R}\left(\frac{1}{T^{1-\lambda-\frac{3}{2}\delta}}\right), \tag{4.29}$$

uniformly with respect to θ in $[0,1]$. Therefore, from (4.25)–(4.29),

$$\begin{aligned}
&\Sigma_{a,b,\delta,s,w,T} \\
&= \sum_{Lm_1\le m<L(m_1+1)} \sum_{Ln_1\le n<L(n_1+1)} \frac{1}{L^{\mathrm{Re}(s)+\mathrm{Re}(w)+1/2} m_1^{\mathrm{Re}(s)+1/4} n_1^{\mathrm{Re}(w)+1/4}}\left(1+\frac{1}{T^{1-\lambda-\delta}}\right) \\
&\quad \times \left(\cos\left(b\sqrt{\frac{m_1}{n_1}}\right) + O_{x,R}\left(\frac{1}{T^{1-\lambda-\frac{3}{2}\delta}}\right)\right) \sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)} - \frac{3}{4}\pi\right) \\
&= \frac{\cos\left(b\sqrt{\frac{m_1}{n_1}}\right)}{L^{\mathrm{Re}(s)+\mathrm{Re}(w)+1/2} m_1^{\mathrm{Re}(s)+1/4} n_1^{\mathrm{Re}(w)+1/4}} \\
&\quad \times \sum_{Lm_1\le m<L(m_1+1)} \sum_{Ln_1\le n<L(n_1+1)} \sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)} - \frac{3}{4}\pi\right)
\end{aligned}$$

$$+\ O_{x,R}\left(\frac{L^{3/2-\mathrm{Re}(s)-\mathrm{Re}(w)}}{m_1^{\mathrm{Re}(s)+1/4}n_1^{\mathrm{Re}(w)+1/4}T^{1-\lambda-\frac{3}{2}\delta}}\right). \tag{4.30}$$

Here,

$$\begin{aligned} m_1^{\mathrm{Re}(s)+1/4}n_1^{\mathrm{Re}(w)+1/4} &\gg T^{(\mathrm{Re}(s)+1/4)(1-\lambda)+(\mathrm{Re}(w)+1/4)(1-\lambda-\delta)} \\ &\gg T^{(1-\lambda)(\mathrm{Re}(s)+\mathrm{Re}(w)+1/2)-(\mathrm{Re}(w)+1/4)\delta}. \end{aligned} \tag{4.31}$$

By (4.30) and (4.31),

$$\begin{aligned} |\Sigma_{m_1,n_1}| \ll &\frac{\sum_{Lm_1\le m<L(m_1+1)}\left|\sum_{Ln_1\le n<L(n_1+1)}\sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)}-\frac{3}{4}\pi\right)\right|}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+1/2-(\mathrm{Re}(w)+1/4)\delta}} \\ &+O_{x,R}\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+3/2-3\lambda-(\mathrm{Re}(w)+\frac{7}{4})\delta}}\right). \end{aligned}$$

4.5. **Short Exponential Sums.** Consider the exponential sum

$$E_{m_1,n_1} := \sum_{Lm_1\le m<L(m_1+1)}\ \sum_{Ln_1\le n<L(n_1+1)} e\left(2\sqrt{xm\left(n+\frac{1}{2}\right)}\right). \tag{4.32}$$

From [9, Equation (7.2)],

$$|\Sigma_{m_1,n_1}| = O\left(\frac{|E_{m_1,n_2}|}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+1/2-(\mathrm{Re}(w)+1/4)\delta}}\right)+O\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+3/2-3\lambda-(\mathrm{Re}(w)+7/4)\delta}}\right). \tag{4.33}$$

Since $T_2T_4 = O\left(T^{2-2\lambda+\delta}\right)$, we see that

$$\begin{aligned} |\Sigma_{a,b,\delta,s,w,T}| = O&\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+1/2-(\mathrm{Re}(w)+1/4)\delta}}\sum_{T_1\le m_1\le T_2}\ \sum_{T_3\le n_1\le T_4}|E_{m_1,n_1}|\right) \\ &+O\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-\lambda-(\mathrm{Re}(w)+11/4)\delta}}\right). \end{aligned} \tag{4.34}$$

From [9, Equation (7.30)],

$$\begin{aligned} |E_{m_1,n_1}| = O_{x,\lambda,\delta}&\left(\min\left(L,\frac{1}{\|\sqrt{xn_1/m_1}\|}\right)\cdot\min\left(L,\frac{1}{\|\sqrt{xm_1/n_1}\|}\right)\right) \\ &+O\left(\frac{1}{T^{1-3\lambda-\delta/2}}\right). \end{aligned} \tag{4.35}$$

The contribution of the second O-term on the right-hand side of (4.35) to (4.34) is

$$O_{x,\lambda,\delta}\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+1/2-(\mathrm{Re}(w)+1/4)\delta}}\frac{T_2T_4}{T^{1-3\lambda-3\delta/2}}\right) = O_{x,\lambda,\delta}\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-\lambda-(\mathrm{Re}(w)+11/4)\delta}}\right). \tag{4.36}$$

Therefore, by (4.34)–(4.36),

$$\begin{aligned} &|\Sigma_{a,b,\delta,s,w,T}| \\ &= O_{x,\lambda,\delta}\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+1/2-(\mathrm{Re}(w)+1/4)\delta}}\sum_{T_1\le m_1\le T_2}\sum_{T_3\le n_1\le T_4}\min\left(L,\frac{1}{\|\sqrt{xn_1/m_1}\|}\right)\right. \\ &\left.\times\min\left(L,\frac{1}{\|\sqrt{xm_1/n_1}\|}\right)\right)+O_{x,\lambda,\delta}\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-\lambda-(\mathrm{Re}(w)+11/4)\delta}}\right). \qquad (4.37)\end{aligned}$$

4.6. **The Case When x Is Not an Integer.** From [9, Equation (8.8)],

$$\begin{aligned} &\min\left(L,\frac{1}{\|\sqrt{xn_1/m_1}\|}\right)\min\left(L,\frac{1}{\|\sqrt{xm_1/n_1}\|}\right) \\ &= O_{x,\delta}\left(T^{\delta}\left(\min\left(L,\frac{1}{\|\sqrt{xn_1/m_1}\|}\right)+\min\left(L,\frac{1}{\|\sqrt{xm_1/n_1}\|}\right)\right)\right). \qquad (4.38)\end{aligned}$$

Inserting (4.38) into the right side of (4.37), we obtain

$$\begin{aligned} |\Sigma_{a,b,\delta,s,w,T}| &= O_{x,\lambda,\delta}\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+1/2-(\mathrm{Re}(w)+5/4)\delta}}\sum_{T_1\le m_1\le T_2}\sum_{T_3\le n_1\le T_4}\min\left(L,\frac{1}{\|\sqrt{xn_1/m_1}\|}\right)\right. \\ &\left.+\min\left(L,\frac{1}{\|\sqrt{xm_1/n_1}\|}\right)\right)+O_{x,\lambda,\delta}\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-\lambda-(\mathrm{Re}(w)+11/4)\delta}}\right). \qquad (4.39)\end{aligned}$$

From [9, Equations (8.18), (8.20)], we have

$$\begin{aligned} |\Sigma_{a,b,\delta,s,w,T}| &= O_{x,\lambda,\delta}\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+1/2-(\mathrm{Re}(w)+5/4)\delta}}\sum_{T_1\le m_2\le T_2}\left(T^{\lambda+\delta/2}+m_1T^{\delta}\log T\right)\right) \\ &+O_{x,\lambda,\delta}\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+1/2-(\mathrm{Re}(w)+5/4)\delta}}\sum_{T_3\le n_1\le T_4}\left(T^{\lambda+\delta/2}+n_1T^{\delta}\log T\right)\right) \\ &+O_{x,\lambda,\delta}\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-\lambda-(\mathrm{Re}(w)+11/4)\delta}}\right) \\ &= O_{x,\lambda,\delta}\left(\frac{T^{1+\delta/2}+T^{2-2\lambda+\delta}\log T}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+1/2-(\mathrm{Re}(w)+5/4)\delta}}\right) \\ &+O_{x,\lambda,\delta}\left(\frac{T^{1+3\delta/2}+T^{2-2\lambda+3\delta}\log T}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+1/2-(\mathrm{Re}(w)+5/4)\delta}}\right) \\ &+O_{x,\lambda,\delta}\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-\lambda-(\mathrm{Re}(w)+11/4)\delta}}\right) \\ &= O_{x,\lambda,\delta}\left(\frac{\log T}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+2\lambda-3/2-(\mathrm{Re}(w)+17/4)\delta}}\right) \\ &+O_{x,\lambda,\delta}\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-\lambda-(\mathrm{Re}(w)+11/4)\delta}}\right). \qquad (4.40)\end{aligned}$$

Choosing $\lambda = \frac{1}{3}$ gives

$$|\Sigma_{a,b,\delta,s,w,T}| = O_{x,\lambda,\delta}\left(\frac{\log T}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-5/6-(\mathrm{Re}(w)+17/4)\delta}}\right). \tag{4.41}$$

Using (4.41) with $T = 2^{r_1+j}$ for each term in in (4.21), we deduce that

$$|S_{7,M_2}(a,\theta,\delta,s,w) - S_{7,M_1}(a,\theta,\delta,s,w)| \tag{4.42}$$

$$= O_{x,\delta}\left(\sum_{j=0}^{r_2-r_1}\frac{\log T}{2^{(r_1+j)(\mathrm{Re}(s)+\mathrm{Re}(w)-5/6-(\mathrm{Re}(w)+17/4)\delta)}}\right) \tag{4.43}$$

$$= O_{x,\delta}\left(\frac{\log T}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-5/6-(\mathrm{Re}(w)+17/4)\delta}}\right), \tag{4.44}$$

uniformly with respect to θ in $[0,1]$. This proves that $S_7(a,\theta,\delta,s,w)$ converges for any (s,w) satisfying $\mathrm{Re}(s)+\mathrm{Re}(w) > \frac{5}{6}$ and the convergence is uniform with respect to θ in $[0,1]$.

4.7. **The Case When x Is an Integer.** For a fixed integer x, we first fix a small real number $\eta > 0$. With η fixed, we then choose $\lambda < 1/2$ such that $1/2 - \lambda \le \eta$, and with η and λ fixed, we then choose δ such that $\delta < 1/2 - \lambda$. Once η, λ, δ are fixed, we start by following the same reduction procedure from the foregoing beginning of the proof, which reduces the problem to the study of the uniform convergence of $S_7(a,\theta,\delta,s,w)$. We again arrive at (4.21). As before, fix j with $1 \le j \le r_2 - r_1$, set $T = 2^{r_1+j}$ and consider the sum $\Sigma_{a,b,\theta,\delta,s,w,T}$ defined in (4.22). We divide the sum $\Sigma_{a,b,\theta,\delta,s,w,T}$ into two parts. Consider in $\mathbb{R}^2$ the rectangle $D(\delta,T) := [T,2T] \times [T^{1-\delta},(2T)^{1+\delta}]$. For each divisor d of x, let $V(x,d,\eta,\delta,T)$ be the region in which all points have slope $[\frac{d^2}{x} - \frac{1}{T^{1/2-\eta}}, \frac{d^2}{x} + \frac{1}{T^{1/2-\eta}}]$ from the origin, i.e.,

$$V(x,d,\eta,\delta,T) = \left\{(m,n) \in D(\delta,T) : \left|\frac{n}{m} - \frac{d^2}{x}\right| \le \frac{1}{T^{1/2-\eta}}\right\}. \tag{4.45}$$

Denote

$$U_1(a,b,\delta,\eta,s,w,T) := \sum_{(m,n)\in D(\delta,T)\setminus\cup_{d|x}V(x,d,\eta,\delta,T)} \frac{\cos\left(b\sqrt{\frac{m}{n}}\right)\sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)} - \frac{3}{4}\pi\right)}{m^{s+1/4}n^{w+1/4}} \tag{4.46}$$

and

$$U_2(a,b,\delta,\eta,s,w,T) := \sum_{d|x}\sum_{(m,n)\in V(x,d,\eta,\delta,T)} \frac{\cos\left(b\sqrt{\frac{m}{n}}\right)\sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)} - \frac{3}{4}\pi\right)}{m^{s+1/4}n^{w+1/4}}. \tag{4.47}$$

Thus,

$$\Sigma_{a,b,\delta,s,w,T} = U_1(a,b,\delta,\eta,s,w,T) + U_2(a,b,\delta,\eta,s,w,T). \tag{4.48}$$

The number of points in $V(x,d,\eta,\delta,T)$ is asymptotic to the area of $V(x,d,\eta,\delta,T)$ which is of the order $T^{3/2+\eta}$. Thus, a trivial bound for $U_2(a,b,\delta,\eta,s,w,T)$ is

$$|U_2(a,b,\delta,\eta,s,w,T)| = O_{x,\eta,\delta}\left(T^{1+\eta-\mathrm{Re}(s)-\mathrm{Re}(w)}\right), \tag{4.49}$$

which is insufficient for our purpose.

4.7.1. ***Estimation for $U_1(a,b,\delta,\eta,s,w,T)$***. We first bound $U_1(a,b,\delta,\eta,s,w,T)$. Subdivide $D(\delta,T)\setminus\cup_{d|x} V(x,d,\eta,\delta,T)$ into squares of size $L\times L$, where $L=\lfloor T^{\lambda}\rfloor$. Let T_1,T_2,T_3 and T_4 be defined as in (4.23). For each $m_1\in\{T_1,T_1+1,\dots,T_2\}$ and $n_1\in\{T_2,T_2+1,\dots,T_4\}$, define Σ_{m_1+1,n_1} by (4.24). Consider the squares $[Lm_1,L(m_1+1))\times[Ln_1,L(n_1+1))$ for which the lower left corner does not belong to any of the trapezoids $V(x,d,\eta,\delta,T)$, i.e.,

$$\frac{n_1}{m_1}\notin\bigcup_{d|x}\left[\frac{d^2}{x}-\frac{1}{T^{1/2-\eta}},\frac{d^2}{x}+\frac{1}{T^{1/2-\eta}}\right]. \tag{4.50}$$

Note that the integral points (m,n) in $D(\delta,T)\setminus\cup_{d|x} V(x,d,\eta,\delta,T)$ that do not belong to the union of squares $[Lm_1,L(m_1+1))\times[Ln_1,L(n_1+1))$ with $m_1\in\{T_1,T_1+1,\dots,T_2\}$ and $n_1\in\{T_2,T_2+1,\dots,T_4\}$ satisfying (4.50) are at a distance $O(L)$ from the boundary of $D(\delta,T)\setminus\cup_{d|x} V(x,d,\eta,\delta,T)$. The contribution on the right side of (4.46) from the points (m,n) that are at distance $O(L)$ from the four edges of the rectangle $D(\delta,T)$ can be bounded by $O(T^{-(\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-\lambda-(\mathrm{Re}(w)+1/4)\delta)})$ as estimated in (4.25). Similarly, the contribution from the points (m,n) in $D(\delta,T)$ that are at distance $O(L)$ from the union of rays from the origin of slopes $\frac{d^2}{x}-\frac{1}{T^{1/2-\eta}}$ and $\frac{d^2}{x}+\frac{1}{T^{1/2-\eta}}$ is $O_x\left(T^{-(\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-\lambda)}\right)$. Hence,

$$\left|U_1(a,b,\delta,\eta,s,w,T)-\sum_{\substack{T_1\le m_1\le T_2\\ T_3\le n_1\le T_4\\ \frac{n_1}{m_1}\notin\cup_{d|x}\left[\frac{d^2}{x}-\frac{1}{T^{1/2-\eta}},\frac{d^2}{x}+\frac{1}{T^{1/2-\eta}}\right]}}\Sigma_{m_1,n_1}\right|$$
$$=O_x\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-\lambda-(\mathrm{Re}(w)+1/4)\delta}}\right). \tag{4.51}$$

Applying (4.33) to each Σ_{m_1,n_1}, we obtain

$$|U_1(a,b,\delta,\eta,s,w,T)|$$
$$=O\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+1/2-(\mathrm{Re}(w)+1/4)\delta}}\sum_{\substack{T_1\le m_1\le T_2\\ T_3\le n_1\le T_4\\ \frac{n_1}{m_1}\notin\cup_{d|x}\left[\frac{d^2}{x}-\frac{1}{T^{1/2-\eta}},\frac{d^2}{x}+\frac{1}{T^{1/2-\eta}}\right]}}E_{m_1,n_1}\right)$$
$$+O_x\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-\lambda-(\mathrm{Re}(w)+11/4)\delta}}\right), \tag{4.52}$$

where E_{m_1,n_1} is defined in (4.32). From (4.35),

$$\begin{aligned}&|U_1(a,b,\delta,\eta,s,w,T)|\\&=O\Bigg\{\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+1/2-(\mathrm{Re}(w)+1/4)\delta}}\sum_{\substack{T_1\le m_1\le T_2\\T_3\le n_1\le T_4\\ \frac{n_1}{m_1}\notin\cup_{d|x}\left[\frac{d^2}{x}-\frac{1}{T^{1/2-\eta}},\frac{d^2}{x}+\frac{1}{T^{1/2-\eta}}\right]}}\\&\quad\times\min\left(T^\lambda,\frac{1}{\|\sqrt{xn_1/m_1}\|}\right)\min\left(T^\lambda,\frac{1}{\|\sqrt{xm_1/n_1}\|}\right)\Bigg\}\\&\quad+O_x\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-\lambda-(\mathrm{Re}(w)+11/4)\delta}}\right).\end{aligned}\tag{4.53}$$

Let

$$\begin{aligned}&\mathcal{B}_1(x,\eta,\lambda,\delta,T):=\Bigg\{(m_1,n_1):T_1\le m_1\le T_2,T_3\le n_1\le T_4,\\&\frac{n_1}{m_1}\notin\cup_{d|x}\left[\frac{d^2}{x}-\frac{1}{T^{1/2-\eta}},\frac{d^2}{x}+\frac{1}{T^{1/2-\eta}}\right],\max\left(\left\|\sqrt{\frac{xn_1}{m_1}}\right\|,\left\|\sqrt{\frac{xm_1}{n_1}}\right\|\right)>\frac{1}{T^\delta}\Bigg\}\end{aligned}\tag{4.54}$$

and

$$\begin{aligned}&\mathcal{B}_2(x,\eta,\lambda,\delta,T):=\Bigg\{(m_1,n_1):T_1\le m_1\le T_2,T_3\le n_1\le T_4,\\&\frac{n_1}{m_1}\notin\cup_{d|x}\left[\frac{d^2}{x}-\frac{1}{T^{1/2-\eta}},\frac{d^2}{x}+\frac{1}{T^{1/2-\eta}}\right],\left\|\sqrt{\frac{xn_1}{m_1}}\right\|\le\frac{1}{T^\delta},\left\|\sqrt{\frac{xm_1}{n_1}}\right\|\le\frac{1}{T^\delta}\Bigg\}.\end{aligned}\tag{4.55}$$

The contribution from $\mathcal{B}_1(x,\eta,\lambda,\delta,T)$ on the right side of (4.53) can be estimated as before in (4.40). Consequently,

$$\begin{aligned}&|U_1(a,b,\delta,\eta,s,w,T)|\\&=O_{x,\lambda,\delta}\Bigg(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+1/2-(\mathrm{Re}(w)+1/4)\delta}}\sum_{(m_1,n_1)\in\mathcal{B}_2(x,\eta,\lambda,\delta,T)}\min\left\{T^\lambda,\frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\\&\quad\times\min\left\{T^\lambda,\frac{1}{\sqrt{xm_1/n_1}}\right\}\Bigg)+O_{x,\lambda,\delta}\left(\frac{\log T}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+2\lambda-3/2-(\mathrm{Re}(w)+17/4)\delta}}\right)\\&\quad+O_{x,\lambda,\delta}\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-\lambda-(\mathrm{Re}(w)+11/4)\delta}}\right).\end{aligned}\tag{4.56}$$

For each $(m_1,n_1)\in\mathcal{B}_2(x,\eta,\lambda,\delta,T)$, from [9, Equation (9.20)], we have

$$\min\left\{T^\lambda,\frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\cdot\min\left\{T^\lambda,\frac{1}{\sqrt{xm_1/n_1}}\right\}=O_x\left(\left(\min\left\{T^\lambda,\frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\right)^2\right).\tag{4.57}$$

Inserting (4.57) into (4.56), we find that

$$\begin{aligned}
&|U_1(a,b,\delta,\eta,s,w,T)|\\
&=O_{x,\lambda,\delta}\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+1/2-(\mathrm{Re}(w)+1/4)\delta}}\sum_{(m_1,n_1)\in\mathcal{B}_2(x,\eta,\lambda,\delta,T)}\left(\min\left\{T^{\lambda},\frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\right)^2\right)\\
&+O_{x,\lambda,\delta}\left(\frac{\log T}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+2\lambda-3/2-(\mathrm{Re}(w)+17/4)\delta}}\right)+O_{x,\lambda,\delta}\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-\lambda-(\mathrm{Re}(w)+11/4)\delta}}\right).
\end{aligned}\tag{4.58}$$

For any $(m_1,n_1)\in\mathcal{B}_2(x,\eta,\lambda,\delta,T)$, from [9, Equation (9.22)],

$$\min\left(T^{\lambda},\frac{1}{\|\sqrt{xn_1/m_1}\|}\right)=\frac{1}{\|\sqrt{xn_1/m_1}\|},\tag{4.59}$$

and from [9, Equation (9.23)],

$$\sum_{(m_1,n_1)\in\mathcal{B}_2(x,\eta,\lambda,\delta,T)}\min\left(T^{\lambda},\frac{1}{\|\sqrt{xn_1/m_1}\|}\right)^2=O_{xm\eta,\delta,\lambda}\left(T^{2-2\eta-\lambda}\right)+O_{x,\eta,\delta,\lambda}\left(T^{\frac{5}{2}-\eta-2\lambda}\right).\tag{4.60}$$

Using (4.60) in (4.58), we find that

$$\begin{aligned}
&|U_1(a,b,\delta,\eta,s,w,T)|\\
&=O_{x,\lambda,\delta,\eta}\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-3/2+\lambda+2\eta-(\mathrm{Re}(w)+1/4)\delta}}\right)+O_{x,\lambda,\delta,\eta}\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-2+2\lambda+\eta-(\mathrm{Re}(w)+1/4)\delta}}\right)\\
&\quad+O_{x,\lambda,\delta}\left(\frac{\log T}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+2\lambda-3/2-(\mathrm{Re}(w)+17/4)\delta}}\right)+O_{x,\lambda,\delta}\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-\lambda-(\mathrm{Re}(w)+11/4)\delta}}\right).
\end{aligned}\tag{4.61}$$

If $\lambda=\frac{1}{2}-\frac{1}{3}\eta$, then, from (4.61),

$$|U_1(a,b,\delta,\eta,s,w,T)|=O_{x,\lambda,\delta,\eta}\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)+\frac{1}{3}\eta-1-(\mathrm{Re}(w)+11/4)\delta}}\right).\tag{4.62}$$

4.7.2. ***Estimation for $U_2(a,b,\delta,\eta,s,w,T)$***. For each $(m,n)\in V(x,d,\eta,T)$, by (4.45), we see that

$$\left|\frac{n}{m}-\frac{d^2}{x}\right|\leq\frac{1}{T^{1/2-\eta}},$$

and thus

$$\cos\left(b\sqrt{\frac{m}{n}}\right)=\cos\left(\frac{b\sqrt{x}}{d}\right)+O_x\left(\frac{1}{T^{1/2-\eta}}\right),\tag{4.63}$$

and

$$\frac{1}{m^{s+1/4}n^{w+1/4}}=\frac{x^{w+1/4}}{d^{2w+1/2}m^{s+w+1/2}}\left(1+O_x\left(\frac{1}{T^{1/2-\eta}}\right)\right).\tag{4.64}$$

Hence, by (4.47),

$$U_2(a,b,\delta,\eta,s,w,T) = x^{w+\frac{1}{4}} \sum_{d|x} \frac{\cos\left(\frac{b\sqrt{x}}{d}\right)}{d^{2w+1/2}} \sum_{(m,n)\in V(x,d,\eta,T)} \frac{\sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)}-\frac{3}{4}\pi\right)}{m^{s+w+1/2}}$$
$$+ O_x\left(\frac{1}{T^{1/2-\eta}} \sum_{d|x} \sum_{(m,n)\in V(x,d,\eta,T)} \frac{1}{m^{\mathrm{Re}(s)+\mathrm{Re}(w)+1/2}}\right). \tag{4.65}$$

The number of integral points $(m,n) \in V(s, d\eta, T)$ is of order $T^{3/2+\eta}$. Thus the O_x term in (4.65) is $O_x\left(T^{-(\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-2\eta)}\right)$. From [9, Equation (10.5)],

$$\sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)}-\frac{3}{4}\pi\right)$$
$$= \sin\left(\frac{\pi x}{d} + \frac{2\pi x n}{d} - \frac{\pi x^2\left(\frac{1}{2}+n-\frac{d^2 m}{x}\right)^2}{2d^3 m} - \frac{3}{4}\pi\right) + O\left(\frac{1}{T^{1/2-3\eta}}\right). \tag{4.66}$$

Thus, (4.65) becomes

$$U_2(a,b,\delta,\eta,s,w,T) = x^{w+1/4} \sum_{d|x} (-1)^{\frac{x}{d}+1} \frac{\cos\left(\frac{b\sqrt{x}}{d}\right)}{d^{2w+1/2}} \sum_{(m,n)\in V(x,d,\eta,T)} \frac{\sin\left(\frac{\pi x^2\left(\frac{1}{2}+n-\frac{d^2 m}{x}\right)^2}{2d^3 m} + \frac{3}{4}\pi\right)}{m^{s+w+1/2}}$$
$$+ O_x\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-4\eta}}\right). \tag{4.67}$$

Furthermore,

$$\frac{\pi x^2\left(\frac{1}{2}+n-\frac{d^2 m}{x}\right)^2}{2d^3 m} = \frac{\pi\left(xn-d^2 m\right)^2}{2d^3 m} + O_x\left(\frac{1}{T^{1/2-\eta}}\right), \tag{4.68}$$

which, when inserted in (4.67), gives

$$U_2(a,b,\delta,\eta,s,w,T) = x^{w+1/4} \sum_{d|x} (-1)^{\frac{x}{d}+1} \frac{\cos\left(\frac{b\sqrt{x}}{d}\right)}{d^{2w+1/2}} \sum_{(m,n)\in V(x,d,\eta,T)} \frac{\sin\left(\frac{\pi\left(xn-d^2 m\right)^2}{2d^3 m} + \frac{3}{4}\pi\right)}{m^{s+w+1/2}}$$
$$+ O_x\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-4\eta}}\right). \tag{4.69}$$

For each divisor d of x, let $H_d(u,v)$ be defined on $[T,2T) \times [T^{1-\delta},(2T)^{1+\delta})$ by

$$H_d(u,v) := \frac{\sin\left(\frac{(\pi(xv-d^2u)^2)}{2d^3 u} + \frac{3}{4}\pi\right)}{u^{s+w+1/2}}. \tag{4.70}$$

Following the argument in [9, Equations (10.10)–(10.17)], by replacing s by $s+w-1/2$, we find that

$$U_2(a,b,\delta,\eta,s,w,T) = x^{w+1/4}\sum_{d|x}(-1)^{\frac{x}{d}+1}\frac{\cos\left(\frac{b\sqrt{x}}{d}\right)}{d^{2w+1/2}}\int\int_{V(s,d,\eta,T)} H_d(u,v)dvdu + O_x\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-4\eta}}\right). \tag{4.71}$$

As in [9, Equations (10.18)–(10.20)],

$$\int\int_{V(x,d,\eta,T)} H_d(u,v)dvdu = \frac{T^{1-s-w}}{x}\int_1^2 \frac{1}{t^{s+w}}\int_{-T^\eta x t^{1/2}d^{-3/2}}^{T^\eta x t^{1/2}d^{-3/2}} \sin\left(\frac{\pi}{2}y^2+\frac{3}{4}\pi\right)dydt. \tag{4.72}$$

Note that

$$\int_{-\infty}^{\infty}\sin\left(\frac{\pi}{2}y^2+\frac{3}{4}\pi\right)dy = \int_{-T^\eta x t^{1/2}d^{-3/2}}^{T^\eta x t^{1/2}d^{-3/2}}\sin\left(\frac{\pi}{2}y^2+\frac{3}{4}\pi\right)dy + O_x\left(\frac{1}{T^\eta}\right); \tag{4.73}$$

thus

$$U_2(a,b,\delta,\eta,s,w,T) = \frac{T^{1-s-w}c_0}{x^{3/4-w}}\int_1^2\frac{dt}{t^{s+w}}\sum_{d|x}(-1)^{\frac{x}{d}+1}\frac{\cos\left(\frac{b\sqrt{x}}{d}\right)}{d^{2w-1}} + O_x\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1+\eta}}\right) + O_x\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-4\eta}}\right). \tag{4.74}$$

From [9, p. 55] or [13, p. 435, formula 3.691, no. 1],

$$c_0 = -\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}\sin\left(\frac{\pi}{2}y^2\right)dy + \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}\cos\left(\frac{\pi}{2}y^2\right)dy = -\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}} = 0. \tag{4.75}$$

Therefore,

$$|U_2(a,b,\delta,\eta,s,w,T)| = O_x\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1+\eta}}\right) + O_x\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-4\eta}}\right). \tag{4.76}$$

Combining (4.62) and (4.76), we have

$$\Sigma_{a,b,\delta,\eta,s,w,T} = O_x\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1+\frac{1}{3}\eta-(\mathrm{Re}(w)+1/2)\delta}}\right) + O_x\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1+\eta}}\right) + O_x\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-1/2-4\eta}}\right). \tag{4.77}$$

Choosing $\eta = \frac{3}{26}$ gives

$$\Sigma_{a,b,\delta,\eta,s,w,T} = O_x\left(\frac{1}{T^{\mathrm{Re}(s)+\mathrm{Re}(w)-25/26}}\right). \tag{4.78}$$

Thus, for $\mathrm{Re}(s)+\mathrm{Re}(w) > \frac{25}{26}$, $\mathrm{Re}(s) > \frac{1}{4}$ and $\mathrm{Re}(w) > \frac{1}{4}$, the series $S_7(a,\theta,\delta,s,w)$ converges uniformly with respect to θ in $[0,1]$.

In conclusion, if either x is not an integer and $\mathrm{Re}(s)+\mathrm{Re}(w) > \frac{5}{6}$, $\mathrm{Re}(s) > \frac{1}{4}$ and $\mathrm{Re}(w) > \frac{1}{4}$, or if x is an integer and $\mathrm{Re}(s) + \mathrm{Re}(w) > \frac{25}{26}$, $\mathrm{Re}(s) > \frac{1}{4}$ and $\mathrm{Re}(w) > \frac{1}{4}$, then the series $G(x, \theta, s, w)$ converges uniformly with respect to θ in any compact subinterval of $(0, 1)$. This completes the proof of Theorem 4.1. □

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