Beatty Sequences, Exotic Number Systems, and Partitions of the Integers

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Introduction

Goals

- Learn integer partitions such as the partitions arising in Beatty's Theorem.
- Investigate "Beatty-type" partitions involving more than two parts.
- Learn relevant background material on Weyl's Theorem and related topics.
- Use Mathematica to explore related questions.

Beatty's Theorem

Theorem. *Let α, β be two positive irrational numbers. Let A and B be two sequences such that $A = (\lfloor n\alpha \rfloor)_n = \{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \ldots\}$, $B = (\lfloor n\beta \rfloor)_n = \{\lfloor \beta \rfloor, \lfloor 2\beta \rfloor, \lfloor 3\beta \rfloor, \ldots\}$. Then A and B form a partition of the integers if and only if*

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

When $\alpha = \phi, \beta = \phi^2$ ($\phi = \frac{\sqrt{5}+1}{2}$ is the golden ratio), $\frac{1}{\phi} + \frac{1}{\phi^2} = 1$, so we have a partition of the integers:

$$A = (\lfloor n\phi \rfloor)_n = \{1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21, \ldots\},$$
$$B = (\lfloor n\phi^2 \rfloor)_n = \{2, 5, 7, 10, 13, 15, 18, 20, 23, 26, 28, \ldots\}.$$

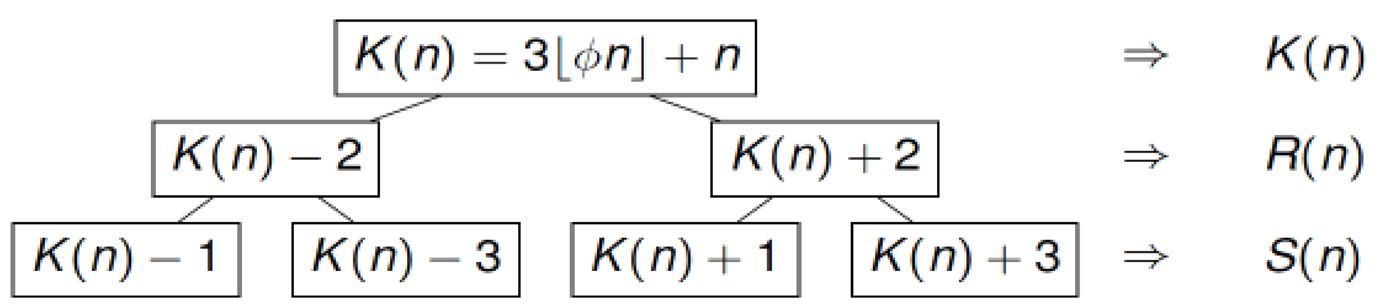
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Uspensky's Theorem

Theorem. *If $k > 2$, then there do not exist k positive real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that the sequences $(\lfloor \alpha_1 n \rfloor)_n, (\lfloor \alpha_2 n \rfloor)_n, \ldots, (\lfloor \alpha_k n \rfloor)_n$ partition the positive integers.*

3-Partition and Almost Beatty Partition

Three Sets Partition Algorithm



$$K(n) = \{4, 8, 12, 16, 21, 25, 29, 33, 38 \ldots\}$$
$$R(n) = \{6, 13, 20, 27, 34, 41, 47, 54, 61 \ldots\}$$
$$S(n) = \{1, 2, 3, 5, 7, 9, 10, 11, 14, 15, 17 \ldots\}$$

Almost Beatty Partitions

Question: How close can the form of this 3-partition be to a "Beatty-type" partition? How close are these sequences to something of the form

$$(\lfloor \alpha n \rfloor)_n, (\lfloor \beta n \rfloor)_n, (\lfloor \gamma n \rfloor)_n$$

where

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1\ ?$$

Let $\alpha = \frac{(\phi+2)}{2}$, $\beta = (\phi + 2)$, $\gamma = (3\phi + 1)$, then $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1$, so according to Uspensky's theorem $\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor, \lfloor \gamma n \rfloor$ will not generate a partition of integers, but we have

$$0 \le \lfloor \alpha n \rfloor - S(n) \le 1,$$
$$0 < \lfloor \beta n \rfloor - R(n) = 1,$$
$$0 \le \lfloor \gamma n \rfloor - K(n) \le 2.$$

Question: How close can a 3-partition be to three Beatty Sequences?

We try to construct a "Beatty-type" 3-partition where two of the columns are exact Beatty sequences and the third one differs from a Beatty sequence by at most 2. Here is an example where $E = (\lfloor n\phi^3 \rfloor)_n, F = (\lfloor n\phi^4 \rfloor)_n$.

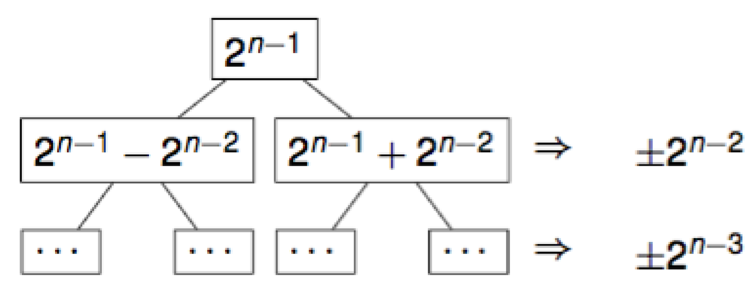
$$E(n) = \{4, 8, 12, 16, 21, 25, 29, 33, 38 \ldots\}$$
$$F(n) = \{6, 13, 20, 27, 34, 41, 47, 54, 61 \ldots\}$$
$$G(n) = \{1, 2, 3, 5, 7, 9, 10, 11, 14, 15, 17 \ldots\}$$

Conjecture: $G(n)$ differs from $(\lfloor n\phi \rfloor)_n$ by at most 2, where

$$(\lfloor n\phi \rfloor)_n = \{1, 3, 4, 6, 8, 9, 11, 12, 14, 16 \ldots\}.$$

More General Partitions

Partition of the Integers into n Parts



Above is a tree which depicts a construction of the $\infty \times n$ matrix which forms a disjoint partition of the positive integers. We take the initial value 2^{n-1} and construct the finite tree above by successively adding and subtracting descending powers of two from the numbers in some level of the tree and adding the results to a new level. This continues until we add and subtract 1 from some level of the tree to form the final level. Each level gives the beginning of an increasing sequence of positive integers. If we denote the first sequence as $d(k)$ and the second as $c(k)$, then we have

$$d(1) = 2^{n-1}, c(1) = 2^{n-1} - 2^{n-2}, c(2) = 2^{n-1} + 2^{n-2}.$$

In general, the partition is formed by constructing infinitely many trees with the $d(k)$ sequence as the roots in this manner and adding the numbers in each level to their corresponding sequences. We define $d(k)$ as follows, in terms of the difference $\Delta_k = c(k) - c(k-1)$:

$$d(k) = \begin{cases} d(k-1) + 2^{n-1} & \text{if } \Delta_k = 2^{n-1} - 1, \\ d(k-1) + 2^n - 1 & \text{if } \Delta_k = 2^{n-1}. \end{cases}$$

Theorem. *The n-th part and the $(n-1)$-th part in the above n-partition are given in the table below, where $A(j) = \lfloor \phi j \rfloor$. This generalizes the Beatty Partition and the 3-Partition shown on the left.*

k	1	...	$(n-1)$-th column, $c(k)$	n-th column, $d(k)$
1	1	...	2^{n-2}	2^{n-1}
2	3	$\cdots$	$2^{n-1} + 2^{n-2}$	2^n
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j	...	...	$A(j) + (2^{n-1} - 2)j - (2^{n-2} - 1)$	$(2^{n-1} - 1)A(j) + j$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Future Directions

Question:

- Can we get a formula for the r-th column in this n partition?
- When $n = 2$, this is a Beatty Partition with $A(n) = \lfloor \phi n \rfloor$ and $B(n) = \lfloor \phi^2 n \rfloor$. When $n = 3$, it is the 3-Partition we constructed, where $K = 3\lfloor \phi n \rfloor + n$. What is the limit of this partition when $n \to \infty$?

Conjecture:
When $n \to \infty$, it is the partition of integers where the n-th part consists of integers of the form $2^{n-1}(2m-1)$ $(1 \le n, m < \infty)$.The columns form a disjoint partition of the positive integers.

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